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FOUR-DIMENSIONAL EINSTEIN METRICS FROM BICONFORMAL DEFORMATIONS

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Abstract. Biconformal deformations take place in the presence of a conformal foliation, deforming by different factors tangent to and orthogonal to the foliation. Four-manifolds endowed with a conformal foliation by surfaces present a natural context to put into effect this process. We develop the tools to calculate the transformation of the Ricci curvature under such deformations and apply our method to construct Einstein 4-manifolds. Examples of one particular family have ends which collapse asymptotically to $\mathbb{R}^2$.

1. Introduction

A smooth Riemannian manifold (M,g) is said to be *Einstein* if its Ricci curvature satisfies $\mathrm{Ric} = Ag$ for some constant A. D. Hilbert showed how Einstein metrics arise from the variational problem of extremizing scalar curvature [8]. The relation between scalar curvature and conformal transformations has been explored by analysts over the latter part of the last century. The Yamabe problem is to determine the existence of a metric of constant scalar curvature in a conformal class [14]. There have been important contributions by various authors and the problem was completely solved positively in the compact case by R. Schoen [10]; for a survey see the notes of Hebey [6].

Conformal transformations are not in general sufficiently discerning to find Einstein metrics. For example, although any manifold admits a Riemannian metric, on a compact manifold, there is a topological obstruction to the existence of an Einstein metric, known as the Hitchin-Thorpe inequality [2, 9, 12], whereas there always exist constant scalar curvature metrics. Biconformal deformations on the other hand, appear optimal to control the Ricci curvature.

A biconformal deformation of a Riemannian manifold (M,g) (see below) takes place in the presence of a conformal foliation. A foliation $\mathcal{F}$ is *conformal* if Lie transport along the leaves of the normal space is conformal [13], specifically, if we

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set $T\mathcal{F}$ to be the tangent space to the leaves and $N\mathcal{F}$ the normal space, there exists a mapping $a\colon T\mathcal{F} \to \mathbb{R}$, linear at each point, such that

$$(\mathcal{L}_U g)(X,Y) = a(U)g(X,Y) \qquad (\forall U \in T\mathcal{F}\,, \forall X, Y \in N\mathcal{F})\,.$$

Conformal foliations are intimately related to semi-conformal mappings.

A mapping $\varphi\colon (M^m, g) \to (N^n, h)$ is *semi-conformal* if at each point where its derivative is non-zero, it is surjective and conformal (and so homothetic) on the complement of its kernel. Specifically, at each $x \in M$ where $\mathrm{d}\varphi_x \neq 0$, the derivative is surjective and there exists a real number $\lambda(x) > 0$ such that

$$\varphi^* h(X,Y) = \lambda(x)^2 g(X,Y) \qquad (\forall X, Y \in (\ker \mathrm{d}\varphi_x)^\perp)\,.$$

Extending λ to be zero at points x where $\mathrm{d}\varphi_x = 0$, determines a continuous function $\lambda\colon M \to \mathbb{R}(\geq 0)$, smooth away from critical points, called the *dilation* of φ. In [1], it is shown that if $\varphi\colon (M^m, g) \to (N^n, h)$ is a semi-conformal submersion, then its fibres form a conformal foliation; conversely, if $\mathcal{F}$ is a conformal foliation on (M^m, g) and $\psi\colon W \subset M \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ is a local foliated chart, then there is a conformal metric on the leaf space N of $\mathcal{F}|_W$ with respect to which the natural projection $\varphi\colon W \to N$ is a semi-conformal submersion. The relation between a above and the dilation λ is given by $a = -2\mathrm{d}\ln\lambda|_{\mathcal{V}}$, where $\mathcal{V} = T\mathcal{F} = \ker \mathrm{d}\varphi$ [1].

Let $\varphi\colon (M^n, g) \to (N^n, h)$ be a semi-conformal submersion between Riemannian manifolds. Then the metric g decomposes into the sum $g = g^H + g^V$ of its horizontal and vertical components. A *biconformal deformation* of g is a metric

$$\widetilde{g} = \frac{g^H}{\sigma^2} + \frac{g^V}{\rho^2}\,,$$

where σ, $\rho\colon M \to \mathbb{R}$ are smooth positive functions. Note that the deformation is conformal if and only if $\sigma \equiv \rho$. We could equally define a biconformal deformation with respect to a conformal foliation. Such deformations preserve semi-conformality of φ.

The idea to use biconformal deformations to construct 4-dimensional Einstein metrics is founded on the possibility of obtaining a suitable expression for the Ricci curvature in terms of parameters of the semi-conformal map: its dilation, second fundamental form of its fibres, integrablity form associated to the horizontal distribution and the almost complex structure J given by rotation through $\pi/2$ in the horizontal and vertical spaces. When the mapping is a harmonic morphism with 1-dimensional fibres, an elegant expression was exploited by L. Danielo to construct Einstein metrics in dimension 4 by biconformally deforming the metric with respect to a harmonic morphism to a 3-manifold, with the deformation restricted to preserve harmonicity [3, 4].

In this article we achieve a computation of the Ricci curvature associated to a semi-conformal submersion $\varphi\colon (M^4, g) \to (N^2, h)$ (see §3) and use it to construct Einstein metrics by biconformal deformation associated to orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^2$. Amongst the examples produced are warped product solutions deriving from a 3-dimensional dynamical system (see §5.1) and a family of complete Einstein metrics of negative Ricci curvature with each member having an $\mathbb{R}^2$-end (Theorem

5.2). The term *end* is used loosely here to refer to a component of the exterior of a family of exhaustive subsets (not compact) that collapses to $\mathbb{R}^2$.

In §2, we calculate the connection coefficients associated to a semi-conformal submersion $\varphi\colon (M^4, g) \to (N^2, h)$. We exploit these formulae in §3 to deduce expressions for the Ricci curvature in terms of the geometric parameters associated to φ referred to above. In §4, we obtain expressions for how these quantities change under biconformal transformation. These are then applied in §5 to orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^2$, to deduce partial differential equations for an Einstein metric in terms of the parameters σ and ρ. In general these are challenging to solve, but special cases yield interesting and possibly new 4-dimensional Einstein metrics.

2. Connection coefficients associated to a semi-conformal submersion

Let $\varphi\colon (M^4, g) \to (N^2, h)$ be a semi-conformal submersion between oriented Riemannian manifolds with dilation $\lambda\colon M \to \mathbb{R}^+$. The coefficients of the Levi-Civita connection with respect to an adapted orthonormal frame field will be expressed in terms of the dilation, the mean-curvature of the fibres and an integrability form associated to the horizontal distribution.

Let $\{f_1, f_2\}$ be a positive orthonormal frame on N^2. Then in general $\nabla f_1 = \rho_{12} f_2$ and $\nabla f_2 = \rho_{21} f_2$ where $\rho_{12} = -\rho_{21}$ is the associated Cartan 1-form. Since the notion of semi-conformal is conformally invariant and since any Riemannian surface is locally conformally equivalent to a domain of $\mathbb{R}^2$ with its standard metric, for the rest of this section, we suppose the frame $\{f_1, f_2\}$ parallel, so the connection form ρ_{12} vanishes. By a trick, we will later remove this assumption in our expression for the Ricci curvature.

Let $\{e_1, e_2, e_3, e_4\}$ be a positive orthonormal frame on M^4 such that $\mathrm{d}\varphi(e_i) = \lambda f_i$ for $i = 1, 2$, and $e_3, e_4 \in V := \ker \mathrm{d}\varphi$. We will use indices in the following way: $i, j, \ldots \in \{1, 2\}$, $r, s, \ldots \in \{3, 4\}$, $a, b, \ldots \in \{1, 2, 3, 4\}$ and sum over repeated indices. At each $x \in M$, let $\mathcal{H}_x : T_x M \to H_x = V_x^{\perp}$ denote orthogonal projection onto the horizontal space. If we don't wish to be specific about the point x we will simply write $\mathcal{H}$. Similarly, $\mathcal{V}$ denotes projection onto the vertical space.

Define complementary indices $i', j', \ldots$ by $i' = 2$ if $i = 1$ and $i' = 1$ if $i = 2$. Set J^H to be rotation by $+\pi/2$ in the horizontal space H, thus: $J^H(e_1) = e_2$ and $J^H(e_2) = -e_1$, equivalently $J^H(e_i) = (-1)^{i+1} e_{i'}$. Similarly, set J^V to be rotation by $+\pi/2$ in the vertical space V, thus: $J^V(e_3) = e_4$ and $J^V(e_4) = -e_3$. Then $J := (J^H, J^V)$ defines an almost Hermitian structure on (M, g).

Definition 2.1. For a semi-conformal submersion as above, define the *integrability 1-form* $\zeta\colon TM \to \mathbb{R}$ by

$$\zeta(X) := g(\nabla_{e_1} e_2, \mathcal{V}(X)) = \tfrac{1}{2} g([e_1, e_2], \mathcal{V}(X)) \qquad \forall\, X \in TM\,,$$

where $\mathcal{V}$ is orthogonal projection onto $\ker \mathrm{d}\varphi$ and the second equality follows from *Lemma 2.4(i)* below. Then, ζ is well-defined independently of the (positive) horizontal orthonormal frame $\{e_1, e_2\}$ and vanishes if and only if the horizontal distribution is integrable.

Definition 2.2. Let $\mathcal{S} = \varphi^{-1}(y)$ be a fibre of φ. Then for vector fields X, Y tangent to $\mathcal{S}$, we have

$$\nabla_X Y = \nabla^{\mathcal{S}}_X Y + B_X Y$$

where ∇ is the connection on M, $\nabla^{\mathcal{S}}$ the connection on $\mathcal{S}$, i.e. $\nabla^{\mathcal{S}}_X Y = \mathcal{V}\nabla_X Y$, and B is the *second fundamental form of* $\mathcal{S}$ (symmetric by integrability of the vertical distribution). Then the *mean curvature of the fibre* $\mu := \frac{1}{2}\operatorname{Tr} B = \frac{1}{2}\mathcal{H}(\nabla_{e_3}e_3 + \nabla_{e_4}e_4)$.

Extend B to all vectors by the formula $B_X Y := \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y$. Then its adjoint is characterized by:

$$g(B_X Y, Z) = g(Y, B^*_X Z) \quad \Rightarrow \quad B^*_X Z = -\mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Z\,.$$

Lemma 2.3 (Fundamental equation of a semi-conformal submersion [1])**.** *For a semi-conformal submersion $\varphi\colon (M^m, g) \to (N^n, h)$, the tension field $\tau_\varphi = \operatorname{Tr}_g \nabla \mathrm{d}\varphi$ is given by*

$$\tau_\varphi = -(n-2)\mathrm{d}\varphi(\operatorname{grad}\ln\lambda) - (m-n)\mathrm{d}\varphi(\mu)$$

where μ is the mean-curvature of the fibres.

Recall that the connection forms ω_{ab} are defined by $\nabla e_a = \sum_b \omega_{ab} e_b$. In order to express the connection coefficients, we require only the form ω_{34}. The following lemma expresses the connection coefficients in terms of the above quantities.

Lemma 2.4.

(i) $\nabla_{e_i} e_j = -e_j(\ln\lambda)e_i + \operatorname{grad}\ln\lambda + (-1)^{i+1}\delta_{ij'}\zeta^\sharp$

(ii) $\nabla_{e_i} e_r = -e_r(\ln\lambda)e_i - \zeta(e_r)Je_i + \omega_{34}(e_i)Je_r$

(iii) $\nabla_{e_r} e_i = -\zeta(e_r)Je_i - B^*_{e_r}e_i$

(iv) $\nabla_{e_r} e_s = B_{e_r}e_s + \omega_{34}(e_r)Je_s$.

Proof. (i) From Lemma 2.3,

$$\tau_\varphi = -2\mathrm{d}\varphi(\mu)\,.$$

But, recalling we sum over repeated indices, $\nabla\mathrm{d}\varphi(e_r, e_r) = -\mathrm{d}\varphi(\nabla_{e_r}e_r) = -2\mathrm{d}\varphi(\mu)$, so that

$$\nabla\mathrm{d}\varphi(e_i, e_i) = \tau_\varphi - \nabla\mathrm{d}\varphi(e_r, e_r) = 0\,.$$

On the other hand,

$$\begin{aligned}\nabla\mathrm{d}\varphi(e_i, e_i) &= \big(-\mathrm{d}\varphi(\nabla_{e_i}e_i) + \nabla^{\varphi^{-1}}_{e_i}\mathrm{d}\varphi(e_i)\big)\\ &= \big(-\mathrm{d}\varphi(\nabla_{e_i}e_i) + e_i(\ln\lambda)\mathrm{d}\varphi(e_i) + \lambda^2\nabla^N_{f_i}f_i\big)\\ &= \big(-\mathrm{d}\varphi(\nabla_{e_i}e_i) + e_i(\ln\lambda)\mathrm{d}\varphi(e_i)\big) = \big(-\mathrm{d}\varphi(\nabla_{e_i}e_i) + e_i(\ln\lambda)\mathrm{d}\varphi(e_i)\big)\,.\end{aligned}$$

The expression for the horizontal component of $\nabla_{e_i}e_j$ now follows when we note that $g(e_1, \nabla e_1) = 0$ etc.

For the vertical component, first note that

$$g([e_r, e_i], e_j) = e_r(\ln\lambda)g(e_i, e_j) \qquad (\forall\ i, j, \in \{1,2\} \quad \forall\ r \in \{3,4\})\,, \tag{1}$$

since, on the one hand

$$\nabla\mathrm{d}\varphi(e_i, e_r) = -\mathrm{d}\varphi(\nabla_{e_i}e_r)\,;$$

on the other hand, by the symmetry of the second fundamental form

$$\begin{aligned}\nabla \mathrm{d}\varphi(e_i, e_r) = \nabla \mathrm{d}\varphi(e_r, e_i) &= -\mathrm{d}\varphi(\nabla_{e_r} e_i) + \nabla^{\varphi^{-1}}_{e_r} \mathrm{d}\varphi(e_i) \\ &= -\mathrm{d}\varphi(\nabla_{e_r} e_i) + e_r(\ln\lambda)\mathrm{d}\varphi(e_i) \\ &\Longrightarrow \mathrm{d}\varphi(\nabla_{e_i} e_r) = \mathrm{d}\varphi(\nabla_{e_r} e_i) - e_r(\ln\lambda)\mathrm{d}\varphi(e_i)\,.\end{aligned}$$

Equation (1) follows. But then

$$\begin{aligned}-g(\nabla_{e_i} e_j, e_r) = g(e_j, \nabla_{e_i} e_r) = g(e_j, \nabla_{e_r} e_i) - e_r(\ln\lambda) g(e_j, e_i) \\ -g(\nabla_{e_j} e_i, e_r) = g(e_i, \nabla_{e_j} e_r) = g(e_i, \nabla_{e_r} e_j) - e_r(\ln\lambda) g(e_i, e_j)\,.\end{aligned}$$

Now add and use the fact that $0 = e_r(g(e_i, e_j)) = g(\nabla_{e_r} e_i, e_j) + g(e_i, \nabla_{e_r} e_j)$.
(ii) follows since

$$\begin{aligned}\mathcal{H}\nabla_{e_i} e_r &= g(\nabla_{e_i} e_r, e_j) e_j = -g(e_r, \nabla_{e_i} e_j) e_j \\ &= -e_r(\ln\lambda) e_i + (-1)^i \zeta(e_r) e_{i'} = -e_r(\ln\lambda) e_i - \zeta(e_r) J^{\mathcal{H}} e_i\,.\end{aligned}$$

(iii) follows from (1) and (ii).
(iv) is a consequence of the definitions. □

Corollary 2.5.

(i) $[e_i, e_j] = e_i(\ln\lambda) e_j - e_j(\ln\lambda) e_i + 2(-1)^{i+1}\delta_{ij'}\zeta^\sharp$

(ii) $[e_r, e_i] = e_r(\ln\lambda) e_i - B^*_{e_r} e_i - \omega_{34}(e_i) J e_r$

(iii) $\nabla_{e_i} e_i = \mathrm{grad}\ \ln\lambda + \mathcal{V}\mathrm{grad}\ \ln\lambda$

(iv) $\nabla_{e_a} e_a = \mathrm{grad}\ \ln\lambda + \mathcal{V}\mathrm{grad}\ \ln\lambda + 2\mu + \omega_{34}(e_r) J e_r\,.$

3. The Ricci curvature

Let $\varphi\colon (M^4, g) \to (N^2, h)$ be a semi-conformal submersion between oriented Riemannian manifolds. Choose an orthonormal frame field $\{e_a\} = \{e_i; e_r\}$ adapted to the horizontal and vertical spaces. The Ricci curvature is determined by its components:

$$\mathrm{Ric} = R_{ab}\theta_a\theta_b = R_{11}{\theta_1}^2 + 2R_{12}\theta_1\theta_2 + \dots$$

where $\{\theta_a\}$ is the dual frame to $\{e_a\}$ and the product $\theta_a\theta_b = \theta_a \odot \theta_b = \frac{1}{2}(\theta_a \otimes \theta_b + \theta_b \otimes \theta_a)$ is the symmetric product of 1-forms. The coefficients R_{ab} are symmetric in their indices and $R_{ab} = \mathrm{Ric}\,(e_a, e_b)$. In order to compute the Ricci curvature associated to a semi-conformal submersion, we will separately calculate the horizontal components R_{ij}, the mixed components R_{ri} and the vertical components R_{rs}.

Define the covariant tensor fields C and C^* by

$$\begin{aligned}C(X, Y) &:= g(B_{e_r} X, B_{e_r} Y) = g(\mathrm{Tr}\,(B^* B)(X), Y) \\ C^*(X, Y) &:= g(B^*_{e_r} X, B^*_{e_r} Y) = g(\mathrm{Tr}\,(B B^*)(X), Y)\,.\end{aligned}$$

Note that C and C^* are well-defined independent of the frame, symmetric and that C vanishes on horizontal vectors and C^* on vertical vectors.

For a general covariant tensor field $T(X, Y, Z, \ldots)$, define its divergence as derivation and contraction with respect to the *first* entry:

$$(\operatorname{div} T)(Y, Z, \ldots) = (\nabla_{e_a} T)(e_a, Y, Z, \ldots) = e_a(T(e_a, Y, Z, \ldots)) - T(\nabla_{e_a} e_a, Y, Z, \ldots) \\ - T(e_a, \nabla_{e_a} Y, Z, \ldots) - T(e_a, Y, \nabla_{e_a} Z, \ldots) - \cdots$$

To the second fundamental form of the fibres B (a $(2,1)$ tensor field), we associate two $(3,0)$-tensor fields. The first of these is $B_1 \colon TM \times TM \times TM \to \mathbb{R}$ determined by

$$B_1(X, Y, Z) = g(X, \mathcal{H}\nabla_{\mathcal{V}Y} \mathcal{V}Z)$$

and the second

$$B_2(X, Y, Z) = g(\mathcal{H}\nabla_{\mathcal{V}X} \mathcal{V}Y, Z)\,.$$

Note that B_1 and B_2 are identical up to ordering of their arguments, however, their divergences differ.

Our aim is to calculate the Ricci curvature in terms of parameters associated to φ. Being a tensorial object, it suffices to calculate Ric at a point x_0 where we can suppose the frame chosen such that $\mathcal{V}\nabla_{e_r} e_s = 0$, for all $r, s = 3, 4$. Such a frame can be constructed by first choosing a local *normal* frame $\{e_r\}$ for the fibre $\varphi^{-1}(\varphi(x_0))$ centered on x_0 (see [11], Vol. 2, Chapter 7) and then extending this to an orthonormal frame $\{e_a\}$ about x_0 in M. In particular, at x_0, we have $\omega_{34}(e_r) = 0$ for $r = 3, 4$.

Lemma 3.1. *Acting on vertical vectors, the divergence of B_1 at x_0 is determined by*

$$(\operatorname{div} B_1)(e_r, e_s) = e_i(g(e_i, B_{e_r} e_s)) - 2\mu^\flat(B_{e_r} e_s) - \mathrm{d}\ln\lambda(B_{e_r} e_s) \\ - g(e_t, \nabla_{e_i} e_r) g(e_i, B_{e_t} e_s) - g(e_t, \nabla_{e_i} e_s) g(e_i, B_{e_r} e_t)$$

(recalling, we sum over repeated indices).

Proof.

$$(\operatorname{div} B_1)(e_r, e_s) = (\nabla_{e_a} B_1)(e_a, e_r, e_s) \\ = e_i(B_1(e_i, e_r, e_s)) - B_1(\nabla_{e_a} e_a, e_r, e_s) - B_1(e_i, \nabla_{e_i} e_r, e_s) \\ - B_1(e_i, e_r, \nabla_{e_i} e_s)\,.$$

From Corollary 2.5(iv), at x_0, $\mathcal{H}\nabla_{e_a} e_a = 2\mu + \mathcal{H}\mathrm{grad}\ln\lambda$; also $\mathcal{V}\nabla_{e_i} e_r = g(e_t, \nabla_{e_i} e_r) e_r$ etc. and the formula follows. $\square$

Lemma 3.2. *Acting on a vertical and a horizontal vector, the divergence of B_2 at x_0 is given by*

$$(\operatorname{div} B_2)(e_r, e_i) = e_s(B_2(e_s, e_r, e_i)) - 2g(B_{\mathcal{V}\mathrm{grad}\ln\lambda} e_r, e_i) - \zeta(\nabla_{e_r} J^{\mathcal{H}} e_i)\,.$$

Proof. Calculating at x_0,

$$(\operatorname{div} B_2)(e_r, e_i) = e_a(B_2(e_a, e_r, e_i)) - B_2(\mathcal{V}\nabla_{e_a} e_a, e_r, e_i) - B_2(e_s, \mathcal{V}\nabla_{e_s} e_r, e_i) \\ - B_2(e_s, e_r, \mathcal{H}\nabla_{e_s} e_i) \\ = e_s(B_2(e_s, e_r, e_i)) - B_2(\mathcal{V}\nabla_{e_j} e_j, e_r, e_i) - B_2(e_s, e_r, \mathcal{H}\nabla_{e_s} e_i)\,.$$

On applying Corollary 2.5(iii) and Lemma 2.4(iii), this becomes

$$e_s(B_2(e_s, e_r, e_i)) - 2g(B_{\mathcal{V}\mathrm{grad}\,\ln\lambda}e_r, e_i) + \zeta(e_s)g(\nabla_{e_r}e_s, J^{\mathcal{H}}e_i)\,.$$

But the latter term equals $-\zeta(e_s)g(e_s, \nabla_{e_r}J^{\mathcal{H}}e_i)$ and the formula follows. $\square$

In what follows, we shall first establish the stated formulae for the case when N^2 is flat; in particular, we can suppose that $\mathrm{d}\varphi(e_i) = \lambda f_i$ where $\{f_i\}$ is a parallel frame: $\nabla f_i = 0$ and apply the formulae of §2. We will then extend the formulae to the case when N^2 is an arbitrary Riemannian surface.

3.1. **The horizontal components of the Ricci curvature.**

First, we require the following lemma.

Lemma 3.3. *The horizontal sectional curvature $K^H := g(R(e_1, e_2)e_2, e_1)$ is given by*

$$K^H = \Delta\ln\lambda - \mathrm{Tr}\,_{\mathcal{V}}\nabla\mathrm{d}\ln\lambda + \|\mathcal{V}\mathrm{grad}\,\ln\lambda\|^2 - 3\|\zeta\|^2\,.$$

Proof. From Lemma 2.4(i) and Corollary 2.5(i),

$$\begin{aligned}
K^H &= g(\nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2, e_1)\\
&= g(\nabla_{e_1}(e_1(\ln\lambda)e_1 + \mathcal{V}\mathrm{grad}\,\ln\lambda) + \nabla_{e_2}(e_2(\ln\lambda)e_1 - \zeta^\sharp), e_1)\\
&\quad - e_1(\ln\lambda)g(\nabla_{e_2}e_2, e_1) + e_2(\ln\lambda)g(\nabla_{e_1}e_2, e_1) - 2g(\nabla_{\zeta^\sharp}e_2, e_1)\\
&= e_1(e_1(\ln\lambda)) + e_2(e_2(\ln\lambda)) - \|\mathcal{V}\mathrm{grad}\,\ln\lambda\|^2 - \|\zeta\|^2\\
&\quad - e_1(\ln\lambda)^2 - e_2(\ln\lambda)^2 - 2\zeta(e_r)g(\nabla_{e_r}e_2, e_1)\\
&= \Delta(\ln\lambda) - \mathrm{Tr}\,_{\mathcal{V}}\nabla\mathrm{d}\ln\lambda + \mathrm{d}\ln\lambda(\nabla_{e_i}e_i) - \|\mathcal{V}\mathrm{grad}\,\ln\lambda\|^2\\
&\quad - \|\mathcal{H}\mathrm{grad}\,\ln\lambda\|^2 - 3\|\zeta\|^2,
\end{aligned}$$

which, from Corollary 2.5(iii), gives the required formula. $\square$

Lemma 3.4. *The horizonal part of the Ricci curvature:* $\mathrm{Ric}\,|_{H\times H}$ *is given by*

$$\mathrm{Ric}\,|_{H\times H} = \left\{\lambda^2K^N + \Delta\ln\lambda + 2\mathrm{d}\ln\lambda(\mu) - 2||\zeta||^2\right\}g^H - C^* + \mathcal{L}_\mu g|_{H\times H}\,,$$

where K^N denotes the Gaussian curvature of N.

Proof. The horizontal components $R_{ij} = \mathrm{Ric}\,(e_i, e_j)$ are given by

$$R_{ij} = g(R(e_i, e_a)e_a, e_j) = K^H g(e_i, e_j) + g(R(e_i, e_r)e_r, e_j)$$

where K^H is given by Lemma 3.3 above.

We now calculate $g(R(e_i, e_r)e_r, e_j) = g(\nabla_{e_i}\nabla_{e_r}e_r - \nabla_{e_r}\nabla_{e_i}e_r - \nabla_{[e_i,e_r]}e_r, e_j)$. Then

$$\begin{aligned}
g(\nabla_{e_i}\nabla_{e_r}e_r, e_j) &= g(\nabla_{e_i}(\mathcal{H}\nabla_{e_r}e_r + \mathcal{V}\nabla_{e_r}e_r), e_j)\\
&= 2g(\nabla_{e_i}\mu, e_j) - g(\mathcal{V}\nabla_{e_r}e_r, \nabla_{e_i}e_j) = 2g(\nabla_{e_i}\mu, e_j)\,.
\end{aligned}$$

From Lemma 2.4(ii) and (iii),

$$\begin{aligned}
-g(\nabla_{e_r}\nabla_{e_i}e_r,e_j) &= -g(\nabla_{e_r}(\mathcal{H}\nabla_{e_i}e_r+\mathcal{V}\nabla_{e_i}e_r),e_j)\\
&= g(\nabla_{e_r}(e_r(\ln\lambda)e_i+\zeta(e_r)Je_i),e_j)-g(\mathcal{V}\nabla_{e_i}e_r,\nabla_{e_r}e_j)\\
&= e_r(e_r(\ln\lambda))g(e_i,e_j)+e_r(\ln\lambda)g(\nabla_{e_r}e_i,e_j)\\
&\quad +g(\nabla_{e_r}(\zeta(e_r)Je_i),e_j)-g(\mathcal{V}\nabla_{e_i}e_r,\nabla_{e_r}e_j)\\
&= \big(\mathrm{Tr}_V\nabla\mathrm{d}\ln\lambda+2\mathrm{d}\ln\lambda(\mu)\big)g(e_i,e_j)\\
&\quad +e_r(\ln\lambda)g(\nabla_{e_r}e_i,e_j)+g\big(\nabla_{e_r}(\zeta(e_r)Je_i),e_j\big)\\
&\quad -g(\mathcal{V}\nabla_{e_i}e_r,\nabla_{e_r}e_j)\,.
\end{aligned}$$

From Lemma 2.4,

$$\begin{aligned}
[e_i,e_r] &= g([e_i,e_r],e_k)e_k+g([e_i,e_r],e_s)e_s\\
&= -e_r(\ln\lambda)e_i+g(\nabla_{e_i}e_r-\nabla_{e_r}e_i,e_s)e_s
\end{aligned}$$

so that

$$\begin{aligned}
-g(\nabla_{[e_i,e_r]}e_r,e_j) &= e_r(\ln\lambda)g(\nabla_{e_i}e_r,e_j)-g(\nabla_{e_i}e_r-\nabla_{e_r}e_i,e_s)g(\nabla_{e_s}e_r,e_j)\\
&= e_r(\ln\lambda)g(-e_r(\ln\lambda)e_i-\zeta(e_r)Je_i,e_j)\\
&\quad -g(\nabla_{e_i}e_r,e_s)g(\nabla_{e_s}e_r,e_j)+g(\nabla_{e_r}e_i,e_s)g(\nabla_{e_s}e_r,e_j)\\
&= -\|\mathcal{V}\mathrm{grad}\,\ln\lambda\|^2g(e_i,e_j)-e_r(\ln\lambda)\zeta(e_r)g(Je_i,e_j)\\
&\quad -g(\nabla_{e_i}e_r,e_s)g(\nabla_{e_s}e_r,e_j)+g(\nabla_{e_r}e_i,e_s)g(\nabla_{e_s}e_r,e_j)\,.
\end{aligned}$$

However, the Ricci tensor is symmetric in its arguments: $\mathrm{Ric}\,(e_i,e_j)=\frac{1}{2}(\mathrm{Ric}\,(e_i,e_j)$ $+\mathrm{Ric}\,(e_j,e_i)$. But then $g(\nabla_{e_i}\mu,e_j)+g(\nabla_{e_j}\mu,e_i)=\mathcal{L}_\mu g(e_i,e_j)$, $g(Je_i,e_j)$ $+g(Je_j,e_i)=0$ and

$$\zeta(e_r)(g(\nabla_{e_r}Je_i,e_j)+g(\nabla_{e_r}Je_j,e_i)=-\zeta(e_r)(g(Je_i,\nabla_{e_r}e_j)+g(Je_i,\nabla_{e_r}e_j)=\|\zeta\|^2\,.$$

Collecting terms now gives the required expression in the case of flat codomain. □

3.2. **The mixed components of the Ricci curvature.**

Lemma 3.5. *For X a horizontal vector and U a vertical vector, one has*

$$\begin{aligned}
\mathrm{Ric}\,(X,U) &= \nabla\mathrm{d}\ln\lambda(X,U)-(\mathrm{d}\ln\lambda)^2(X,U)-2(\mathrm{d}\ln\lambda\odot\zeta)(JX,U)\\
&\quad -(\nabla_{JX}\zeta)(U)-2\zeta(\nabla_UJX)-\mathrm{div}\,B_2(U,X)\\
&\quad -2\mathrm{d}\ln\lambda(B_U^*X)+2(\nabla_U\mu^\flat)(X)\,.
\end{aligned}$$

Proof. By tensoriality, it suffices to set $X=e_i$ and $U=e_r$. Then

$$\mathrm{Ric}\,(e_i,e_r)=g(R(e_i,e_a)e_a,e_r)=g(R(e_i,e_j)e_j,e_r)+g(R(e_r,e_s)e_s,e_i)\,.$$

First, we deal term by term with

$$g(R(e_i,e_j)e_j,e_r)=g(\nabla_{e_i}\nabla_{e_j}e_j-\nabla_{e_j}\nabla_{e_i}e_j-\nabla_{[e_i,e_j]}e_j,e_r)\,.$$

From Corollary 2.5(iii) and Lemma 2.4(ii),

$$\begin{aligned} g(\nabla_{e_i}\nabla_{e_j}e_j, e_r) &= g(\nabla_{e_i}(2\mathrm{grad}\ \ln\lambda - \mathcal{H}\mathrm{grad}\ \ln\lambda), e_r) \\ &= 2\nabla\mathrm{d}\ln\lambda(e_i, e_r) + g(\mathcal{H}\mathrm{grad}\ \ln\lambda, \nabla_{e_i}e_r) \\ &= 2\nabla\mathrm{d}\ln\lambda(e_i, e_r) - e_i(\ln\lambda)e_r(\ln\lambda) - \zeta(e_r)(Je_i)(\ln\lambda)\,. \end{aligned}$$

Also, from Lemma 2.4(ii),

$$\begin{aligned} -g(\nabla_{e_j}\nabla_{e_i}e_j, e_r) &= -e_j(g(\nabla_{e_i}e_j, e_r) + g(\nabla_{e_i}e_j, \nabla_{e_j}e_r) \\ &= e_j(g(e_j, \nabla_{e_i}e_r)) + g(\nabla_{e_i}e_j, \nabla_{e_j}e_r) \\ &= e_j\big(-e_r(\ln\lambda)\delta_{ij} - \zeta(e_r)g(e_j, Je_i)\big) + g(\nabla_{e_i}e_j, \nabla_{e_j}e_r) \\ &= -e_i(e_r(\ln\lambda)) - (Je_i)(\zeta(e_r)) + g(\nabla_{e_i}e_j, \nabla_{e_j}e_r) \\ &= -e_i(e_r(\ln\lambda)) - (\nabla_{Je_i}\zeta)(e_r) - \zeta(\nabla_{Je_i}e_r) + g(\nabla_{e_i}e_j, \nabla_{e_j}e_r)\,, \end{aligned}$$

where, from Lemma 2.4,

$$\begin{aligned} g(\nabla_{e_i}e_j, \nabla_{e_j}e_r) &= g(\nabla_{e_i}e_j, e_k)g(e_k, \nabla_{e_j}e_r) + g(\nabla_{e_i}e_j, e_s)g(e_s, \nabla_{e_j}e_r) \\ &= e_k(\ln\lambda)\delta_{ij}g(e_k, \nabla_{e_j}e_r) - e_j(\ln\lambda)\delta_{ik}g(e_k, \nabla_{e_j}e_r) \\ &\quad + e_s(\ln\lambda)\delta_{ij}g(e_s, \nabla_{e_j}e_r) + (-1)^{i+1}\delta_{ij'}\zeta(e_s)g(e_s, \nabla_{e_j}e_r) \\ &= g(\mathcal{H}\mathrm{grad}\ \ln\lambda, \nabla_{e_i}e_r) - g(e_i, \nabla_{e_j}e_r)e_j(\ln\lambda) \\ &\quad + g(\mathcal{V}\mathrm{grad}\ \ln\lambda, \nabla_{e_i}e_r) + \zeta(e_s)g(e_s, \nabla_{Je_i}e_r) \\ &= -2\zeta(e_r)\mathrm{d}\ln\lambda(Je_i) + \mathrm{d}\ln\lambda(\mathcal{V}\nabla_{e_i}e_r) + \zeta(\nabla_{Je_i}e_r)\,. \end{aligned}$$

From Corollary 2.5(i) and Lemma 2.4(ii),

$$\begin{aligned} -g(\nabla_{[e_i,e_j]}e_j, e_r) &= -e_i(\ln\lambda)\delta_{jk}g(\nabla_{e_k}e_j, e_r) + e_j(\ln\lambda)\delta_{ik}g(\nabla_{e_k}e_j, e_r) \\ &\quad + 2(-1)^i\delta_{ij'}\zeta(e_s)g(\nabla_{e_s}e_j, e_r) \\ &= -2e_i(\ln\lambda)e_r(\ln\lambda) - g(\mathcal{H}\mathrm{grad}\ \ln\lambda, \nabla_{e_i}e_r) \\ &\quad - 2\zeta(e_s)g(\nabla_{e_s}Je_i, e_r) \\ &= -e_i(\ln\lambda)e_r(\ln\lambda) + \zeta(e_r)\mathrm{d}\ln\lambda(Je_i) - 2\zeta(\nabla_{e_r}Je_i)\,. \end{aligned}$$

Collecting terms now yields

$$\begin{aligned} g(R(e_i, e_j)e_j, e_r) &= \nabla\mathrm{d}\lambda(e_i, e_r) - e_i(\ln\lambda)e_r(\ln\lambda) - \zeta(e_r)\mathrm{d}\ln\lambda(Je_i) \\ &\quad - (\nabla_{Je_i}\zeta)(e_r) - 2\zeta(\nabla_{e_r}Je_i)\,. \end{aligned}$$

For the other term, first note that at the point x_0,

$$\begin{aligned} g(\nabla_{e_r}\nabla_{e_s}e_s, e_i) &= g(\nabla_{e_r}(\mathcal{H}\nabla_{e_s}e_s + \mathcal{V}\nabla_{e_s}e_s), e_i) = 2g(\nabla_{e_r}\mu, e_i) \\ &\quad - g(\mathcal{V}\nabla_{e_s}e_s, \nabla_{e_r}e_i) = 2g(\nabla_{e_r}\mu, e_i)\,. \end{aligned}$$

Then from Lemma 3.2,

$$\begin{aligned} g(R(e_r,e_s)e_s,e_i) &= g(\nabla_{e_r}\nabla_{e_s}e_s - \nabla_{e_s}\nabla_{e_r}e_s - \nabla_{[e_r,e_s]}e_s, e_i) \\ &= 2g(\nabla_{e_r}\mu, e_i) - e_s(g(\nabla_{e_r}e_s, e_i)) + g(\nabla_{e_r}e_s, \nabla_{e_s}e_i) \\ &\quad - g(\nabla_{[e_r,e_s]}e_s, e_i) \\ &= 2(\nabla_{e_r}\mu^\flat)(e_i) - (\operatorname{div} B_2)(e_r,e_i) - 2g(\nabla_{e_r}\mathcal{V}\mathrm{grad}\,\ln\lambda, e_i) \\ &\quad - \zeta(\nabla_{e_r}Je_i) + g(\mathcal{H}\nabla_{e_r}e_s, \mathcal{H}\nabla_{e_r}e_i)\,. \end{aligned}$$

But from from Lemma 2.4, $g(\mathcal{H}\nabla_{e_r}e_s, \mathcal{H}\nabla_{e_r}e_i) = -g(\nabla_{e_r}e_s, \zeta(e_s)Je_i) = \zeta(\nabla_{e_r}Je_i)$. The formula now follows for flat codomain. □

3.3. The vertical components of the Ricci curvature.

Define the vertical sectional curvature by $K^V := g(R^F(e_3,e_4)e_4,e_3)$ where $F = \varphi^{-1}(y) \subset M$ is the fibre over $y \in N$ and R^F is the Riemannian curvature of F. Then K^V is related to the sectional curvature in M via the Gauss equation (see [11] Chapter 7):

$$g(R(e_3,e_4)e_4,e_3) = g(R^F(e_3,e_4)e_4,e_3) + |B_{e_3}e_4|^2 - g(B_{e_3}e_3, B_{e_4}e_4)\,.$$

The correction terms have an invariant expression given by the following lemma, established by evaluating the right-hand and left-hand sides on the various (e_r,e_s).

Lemma 3.6.

$$(|B_{e_3}e_4|^2 - g(B_{e_3}e_3, B_{e_4}e_4))g^V = C - 2\mu^\flat(B_\star\star)\,.$$

Lemma 3.7.

$$\mathrm{Ric}\,|_{V\times V} = K^V g^V + 2\nabla\mathrm{d}\ln\lambda|_{V\times V} + 2\mathrm{d}\ln\lambda(B_\star\star) - 2(\mathrm{d}\ln\lambda)^2|_{V\times V} + 2\zeta^2 + \operatorname{div} B_1|_{V\times V}\,.$$

Proof.

$$\begin{aligned} \mathrm{Ric}\,(e_r,e_s) = g(R(e_r,e_a)e_a,e_s) &= (K^V + |B_{e_3}e_4|^2 \\ &\quad - g(B_{e_3}e_3, B_{e_4}e_4))g(e_r,e_s) + g(R(e_r,e_i)e_i,e_s)\,, \end{aligned}$$

with

$$g(R(e_r,e_i)e_i,e_s) = g(\nabla_{e_r}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_r}e_i - \nabla_{[e_r,e_i]}e_i, e_s)\,.$$

From Corollary 2.5(iii), $\nabla_{e_i}e_i = \mathrm{grad}\,\ln\lambda + \mathcal{V}\mathrm{grad}\,\ln\lambda = 2\mathrm{grad}\,\ln\lambda - \mathcal{H}\mathrm{grad}\,\ln\lambda$, so that

$$\begin{aligned} g(\nabla_{e_r}\nabla_{e_i}e_i, e_s) &= 2g(\nabla_{e_r}\mathrm{grad}\,\ln\lambda, e_s) + g(\mathcal{H}\mathrm{grad}\,\ln\lambda, \nabla_{e_r}e_s) \\ &= 2\nabla\mathrm{d}\ln\lambda(e_r,e_s) + \mathrm{d}\ln\lambda(B_{e_r}e_s)\,. \end{aligned}$$

From Lemma 3.1,

$$\begin{aligned} -g(\nabla_{e_i}\nabla_{e_r}e_i, e_s) &= -e_i(g(\nabla_{e_r}e_i, e_s) + g(\nabla_{e_r}e_i, \nabla_{e_i}e_s) \\ &= \operatorname{div} B_1(e_r,e_s) + 2\mu^\flat(B_{e_r}e_s) + \mathrm{d}\ln\lambda(B_{e_r}e_s) \\ &\quad + g(\nabla_{e_r}e_i, e_j)g(e_j, \nabla_{e_i}e_s) + g(\nabla_{e_r}e_i, e_t)g(e_t, \nabla_{e_i}e_s) \\ &= \operatorname{div} B_1(e_r,e_s) + 2\mu^\flat(B_{e_r}e_s) + \mathrm{d}\ln\lambda(B_{e_r}e_s) \\ &\quad + g(e_t, \nabla_{e_i}e_r)g(e_i, \nabla_{e_t}e_s) + g(\nabla_{e_r}e_i, e_j)g(e_j, \nabla_{e_i}e_s)\,, \end{aligned}$$

where the last term can be expressed using Lemma 2.4(ii) and (iii):

$$g(\nabla_{e_r} e_i, e_j) g(e_j, \nabla_{e_i} e_s) = 2\zeta(e_r)\zeta(e_s)\,.$$

From Corollary 2.5(ii) and (iii)

$$\begin{aligned} -g(\nabla_{[e_r,e_i]} e_i, e_s) = & -2e_r(\ln\lambda) e_s(\ln\lambda) - g(e_i, B_{e_r} e_t) g(e_i, B_{e_t} e_s) \\ & + g(e_t, \nabla_{e_i} e_r) g(\nabla_{e_t} e_i, e_s)\,. \end{aligned}$$

On collecting terms and applying Lemma 3.6, the formula follows for the case of flat codomain. □

3.4. **Mapping into an arbitrary curved surface.**

Suppose $\varphi\colon (M^4, g) \to (N^2, h)$ is a semi-conformal submersion into an arbitrary Riemannian surface with dilation λ. About a point in the image of φ, choose local isothermal coordinates $\psi\colon W \to \mathbb{R}^2$ on an open set $W \subset N^2$, so that $h = \nu^{-2}({\mathrm{d}y_1}^2 + {\mathrm{d}y_2}^2)$ for some function $\nu\colon W \to \mathbb{R}$. Consider the following composition:

$$(M^4, g) \xrightarrow{\varphi} (W \subset N^2, h) \xrightarrow{\psi} (W' \subset \mathbb{R}^2, \overline{h})$$

where $\overline{h}$ is the canonical metric ${\mathrm{d}y_1}^2 + {\mathrm{d}y_2}^2$ on $\mathbb{R}^2$ and $W' = \psi(W)$. Then the formulae of §3.1, §3.2 and §3.3 apply to $\psi \circ \varphi$. We now show how they extend to φ.

Lemma 3.8.

$$\lambda^2 K^N \circ \varphi = \Delta \ln(\nu \circ \varphi) + 2\mathrm{d}\ln(\nu \circ \varphi)(\mu)\,.$$

Proof. First note that $K^N = \nu^{-2}\Delta_{\overline{h}} \ln\nu = \Delta_h \ln\nu$. Then from Lemma 2.3,

$$\begin{aligned} \Delta_g(\ln\nu \circ \varphi) &= \mathrm{d}\ln\nu(\tau_\varphi) + \mathrm{Tr}\,_g \nabla\mathrm{d}\ln\nu(\mathrm{d}\varphi, \mathrm{d}\varphi) \\ &= -2\mathrm{d}(\ln\nu \circ \varphi)(\mu) + \lambda^2(\Delta_h \ln\tau) \circ \varphi \\ &= -2\mathrm{d}(\ln\nu \circ \varphi)(\mu) + \lambda^2 K^N \circ \varphi\,. \end{aligned}$$ □

Since the dilation of $\psi \circ \varphi$ is given by $\lambda\nu$, from Lemma 3.4 (for the flat case),

$$\mathrm{Ric}\,|_{H\times H} = \left\{\Delta\ln(\lambda\nu) + 2\mathrm{d}\ln(\lambda\nu)(\mu) - 2||\zeta||^2\right\} g^{\mathcal{H}} - C^* + \mathcal{L}_\mu g|_{H\times H}\,.$$

But from Lemma 3.8,

$$\Delta\ln(\lambda\nu) + 2\mathrm{d}\ln(\lambda\nu)(\mu) = \lambda^2 K^N + \Delta\ln\lambda + 2\mathrm{d}\ln\lambda(\mu)\,,$$

where the latter quantity is invariant with respect to conformal changes of metric on the codomain.

For the mixed components of the Ricci curvature, we note that on setting $\overline{\lambda} = \lambda\nu$,

$$\begin{aligned} &\nabla\mathrm{d}\ln\lambda(X,U) - (\mathrm{d}\ln\lambda)^2(X,U) - 2(\mathrm{d}\ln\lambda \odot \zeta)(JX,U) \\ =&\nabla\mathrm{d}\ln\overline{\lambda}(X,U) - (\mathrm{d}\ln\overline{\lambda})^2(X,U) - 2(\mathrm{d}\ln\overline{\lambda} \odot \zeta)(JX,U)\,. \end{aligned}$$

For example

$$\nabla\mathrm{d}\ln\overline{\lambda}(e_1, e_3) = \nabla\mathrm{d}\ln\lambda(e_1, e_3) - \mathrm{d}\ln(\nu \circ \varphi)(\nabla_{e_1} e_3)\,.$$

But from Lemma 2.4,

$$\begin{aligned}-\mathrm{d}\ln(\nu\circ\varphi)(\nabla_{e_1}e_3) &= -\mathrm{d}\ln(\nu\circ\varphi)(\mathcal{H}\nabla_{e_1}e_3)\\ &= \mathrm{d}\ln(\nu\circ\varphi)(g(e_3,\nabla_{e_1}e_1)e_1+g(e_3,\nabla_{e_1}e_2)e_2)\\ &= 2(\mathrm{d}\ln\lambda\odot\mathrm{d}\ln(\nu\circ\varphi))(e_1,e_3)+2(\mathrm{d}\ln(\nu\circ\varphi)\odot\zeta)(Je_1,e_3)\,.\end{aligned}$$

Whereas

$$\begin{aligned}&-(\mathrm{d}\ln\overline{\lambda})^2(e_1,e_3)-2(\mathrm{d}\ln\overline{\lambda}\odot\zeta)(Je_1,e_3)\\ &\quad=-\,(\mathrm{d}\ln\lambda)^2(e_1,e_3)-2(\mathrm{d}\ln\lambda\odot\zeta)(Je_1,e_3)\\ &\qquad-2(\mathrm{d}\ln\lambda\odot\mathrm{d}\ln(\alpha\circ\varphi))(e_1,e_3)-2(\mathrm{d}\ln(\alpha\circ\varphi)\odot\zeta)(Je_1,e_3)\,.\end{aligned}$$

The invariance of the vertical components of the Ricci curvature follows from the invariance of the quantity $\nabla\mathrm{d}\ln\lambda|_{\mathcal{V}\times\mathcal{V}}+\mathrm{d}\ln\lambda(B_\star\star)$, specifically $\nabla\mathrm{d}\ln\lambda(e_r,e_s)+\mathrm{d}\ln\lambda(B_{e_r}e_s)=e_r(e_s(\ln\lambda))-\mathrm{d}\ln\lambda(\mathcal{V}\nabla_{e_r}e_s)=e_r(e_s(\ln\overline{\lambda}))-\mathrm{d}\ln\overline{\lambda}(\mathcal{V}\nabla_{e_r}e_s)$.

4. Biconformal deformations

4.1. **The effect of a biconformal deformation on the Ricci curvature.** Let $\varphi\colon(M^4,g_0)\to(N^2,h)$ be a semi-conformal map between oriented manifolds. Consider a biconformal deformation:

$$g=\frac{g_0^H}{\sigma^2}+\frac{g_0^V}{\rho^2}$$

where σ, $\rho\colon M^4\to\mathbb{R}$ are smooth strictly positive functions. Write objects with respect to g_0 with an index 0, either upstairs or downstairs, and objects with respect to g as before. For example, the positive orthonormal basis with respect to g_0 will be written $\{e_1^0,e_2^0,e_3^0,e_4^0\}$ and the dilation of φ with respect to g_0 as λ_0, etc. Then the new frame field and the dual field of 1-forms are given by

$$\begin{aligned}&e_1=\sigma e_1^0,\ e_2=\sigma e_2^0,\ e_3=\rho e_3^0,\ e_4=\rho e_4^0\\ &\theta_1=\frac{1}{\sigma}\theta_1^0,\ \theta_2=\frac{1}{\sigma}\theta_2^0,\ \theta_3=\frac{1}{\rho}\theta_3^0,\ \theta_4=\frac{1}{\rho}\theta_4^0\,.\end{aligned}$$

The following lemma gives the change in the connection coefficients.

Lemma 4.1. (i) $g(\nabla_{e_r}e_s,e_i)=g_0(\nabla^0_{e_r^0}e_s^0,e_i)+e_i(\ln\rho)\delta_{rs}$

(ii) $g(\nabla_{e_i}e_r,e_s)=g_0(\nabla^0_{e_i}e_r^0,e_s^0)$

(iii) $g(\nabla_{e_r}e_i,e_j)=g_0(\nabla^0_{e_r}e_i^0,e_j^0)+\frac{\rho^2-\sigma^2}{2\rho^2}g_0([e_i^0,e_j^0],e_r)$

(iv) $g(\nabla_{e_r}e_s,e_t)=g_0(\nabla^0_{e_r^0}e_s^0,e_t)+e_t(\ln\rho)\delta_{rs}-e_s(\ln\rho)\delta_{rt}$

(v) $g(e_i,\nabla_{e_k}e_j)=\sigma g_0(e_i^0,\nabla^0_{e_k^0}e_j^0)+\sigma(e_i^0(\ln\sigma)\delta_{jk}-e_j^0(\ln\sigma)\delta_{ik})$

(vi) $g(\nabla_{e_i}e_j,e_r)=\frac{\sigma^2}{\rho^2}g_0(\nabla^0_{e_i^0}e_j^0,e_r)+\left(1-\frac{\sigma^2}{\rho^2}\right)e_r(\ln\lambda_0)\delta_{ij}+e_r(\ln\sigma)\delta_{ij}\,.$

Proof.

$$\begin{aligned}
\text{(i)}\quad 2g(\nabla_{e_r}e_s,e_i) &= g([e_i,e_r],e_s)+g([e_i,e_s],e_r)\\
&= \sigma g_0([e_i^0,e_r^0],e_s^0)+\sigma g_0([e_i^0,e_s^0],e_r^0)+2\sigma e_i^0(\ln\rho)\delta_{rs}\\
&= 2\sigma g_0(\nabla^0_{e_r^0}e_s^0,e_i^0)+2\sigma e_i^0(\ln\rho)\delta_{rs}\\
\text{(ii)}\quad 2g(\nabla_{e_i}e_r,e_s) &= g([e_i,e_r],e_s)-g([e_i,e_s],e_r)\\
&= \tfrac{1}{\rho^2}g_0([e_i,\rho e_r^0],\rho e_s^0)-\tfrac{1}{\rho^2}g_0([e_i,\rho e_s^0],\rho e_r^0)\\
&= g_0([e_i,e_r^0],e_s^0)-g_0([e_i,e_s^0],e_r^0)+e_i(\ln\rho)\delta_{rs}-e_i(\ln\rho)\delta_{rs}\\
&= 2g_0(\nabla_{e_i}e_r^0,e_s^0)\\
\text{(iii)}\quad 2g(\nabla_{e_r}e_i,e_j) &= g([e_r,e_i],e_j)-g([e_i,e_j],e_r)+g([e_j,e_r],e_i)\\
&= \tfrac{1}{\sigma}g_0([\rho e_r^0,e_i^0],e_j^0)-\tfrac{1}{\rho}g_0([\sigma e_i^0,\sigma e_j^0],e_r^0)+\tfrac{1}{\sigma}g_0([\sigma e_j^0,\rho e_r^0],e_i^0)\\
&= \rho g_0([e_r^0,e_i^0],e_j^0)+\tfrac{\rho}{\sigma}e_r^0(\sigma)\delta_{ij}-\tfrac{\sigma^2}{\rho}g_0([e_i^0,e_j^0],e_r^0)\\
&\quad +\rho g_0([e_j^0,e_r^0],e_i^0)-\tfrac{\rho}{\sigma}e_r^0(\sigma)\delta_{ij}\\
&= 2\rho g_0(\nabla^0_{e_r^0}e_i^0,e_j^0)+\tfrac{\rho^2-\sigma^2}{\rho}g_0([e_i^0,e_j^0],e_r^0)
\end{aligned}$$

(iv) As above, we write $2g(\nabla_{e_r}e_s,e_t)=g([e_r,e_s],e_t)-g([e_s,e_t],e_r)+g([e_t,e_r],e_s)$ and replace e_r by ρe_r^0 etc. Case (v) is similar.

$$\begin{aligned}
\text{(vi)}\quad 2g(\nabla_{e_i}e_j,e_r) &= g([e_i,e_j],e_r)-g([e_j,e_r],e_i)+g([e_r,e_i],e_j)\\
&= \tfrac{1}{\rho}g_0([\sigma e_i^0,\sigma e_j^0],e_r^0)-\tfrac{1}{\sigma}g_0([\sigma e_j^0,\rho e_r^0],e_i^0)+\tfrac{1}{\sigma}g_0([\rho e_r^0,\sigma e_i^0],e_j^0)\\
&= 2\tfrac{\sigma^2}{\rho}g_0(\nabla^0_{e_i^0}e_j^0,e_r^0)+\tfrac{\sigma^2}{\rho}\left(g_0([e_j^0,e_r^0],e_i^0)-g_0([e_r^0,e_i^0],e_j^0)\right)\\
&\quad -\rho g_0([e_j^0,e_r^0],e_i^0)+\rho g_0([e_r^0,e_i^0],e_j^0)+\rho e_r^0(\ln\sigma)\delta_{ij}\\
&\quad +\rho e_r^0(\ln\sigma)\delta_{ij}
\end{aligned}$$

From Lemma 2.4, this gives

$$2g(\nabla_{e_i}e_j,e_r)=2\frac{\sigma^2}{\rho}g_0(\nabla^0_{e_i^0}e_j^0,e_r^0)-2\frac{\sigma^2}{\rho}e_r^0(\ln\lambda_0)\delta_{ij}+2\rho e_r^0(\ln\lambda_0)\delta_{ij}+2\rho e_r^0(\ln\sigma)\delta_{ij}$$

and the formula follows. □

Corollary 4.2.

$$\begin{aligned}
\text{(i)}\quad \nabla_{e_s}e_j &= \sigma\rho\nabla^0_{e_s^0}e_j^0+\tfrac{\rho^2-\sigma^2}{2\rho^2}\zeta^0(e_s)Je_j-e_j(\ln\rho)e_s\\
\text{(ii)}\quad \nabla_{e_r}e_s &= \sigma^2\mathcal{H}\nabla^0_{e_r^0}e_s^0+\rho^2\mathcal{V}\nabla^0_{e_r^0}e_s^0+\delta_{rs}\left(\sigma^2\mathcal{H}\mathrm{grad}_{g_0}\ln\rho+\rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\right)\\
&\quad -\rho^2e_s^0(\ln\rho)e_r^0.
\end{aligned}$$

Proof. From Lemma 4.1,

$$\begin{aligned}
\nabla_{e_s}e_j &= g(\nabla_{e_s}e_j,e_i)e_i+g(\nabla_{e_s}e_j,e_r)e_r=g(\nabla_{e_s}e_j,e_i)e_i-g(e_j,\nabla_{e_s}e_r)e_r\\
&= g_0(\nabla^0_{e_s}e_j^0,e_i^0)e_i+\frac{\rho^2-\sigma^2}{2\rho^2}g_0([e_j^0,e_i^0],e_s)e_i-g_0(\nabla^0_{e_s^0}e_r^0,e_j)e_r-e_j(\ln\rho)\delta_{rs}e_r\\
&= \sigma\rho\nabla^0_{e_s^0}e_j^0+\frac{\rho^2-\sigma^2}{2\rho^2}\zeta^0(e_s)Je_j-e_j(\ln\rho)e_s\,.
\end{aligned}$$

The proof of (ii) is similar. □

Corollary 4.3.

$$B_{e_r}e_s=\sigma^2(B^0_{e_r^0}e_s^0+g_0(e_r^0,e_s^0)\mathcal{H}\mathrm{grad}_{g_0}\ln\rho)\,.$$

Proof. From (i) of Lemma 4.1,

$$g_0(B_{e_r}e_s, e_i^0) = \sigma^2(g_0(\nabla^0_{e_r^0}e_s^0, e_i^0) + e_i^0(\ln\rho)\delta_{rs})$$

from which the formula follows. □

Lemma 4.4. *The mean curvature of the fibres, the integrability form and the dilation change according to*

$$\mu = \sigma^2(\mu_0 + \mathcal{H}\mathrm{grad}_{g_0}\ln\rho), \qquad \zeta = \frac{\sigma^2}{\rho^2}\zeta_0, \qquad \lambda = \sigma\lambda_0\,.$$

Proof. The expression for μ follows by taking the trace in Corollary 4.3. The Lie bracket is defined independently of the metric and the change in ζ follows. The expression for λ follows since the new horizontal basis is a multiple of σ times the old. □

Lemma 4.5. *For a smooth function f,*

$$\mathrm{grad}_g f = \sigma^2\mathrm{grad}_{g_0}f + (\rho^2 - \sigma^2)\mathcal{V}\mathrm{grad}_{g_0}f\,.$$

Proof.

$$\mathrm{grad}_g f = e_a(f)e_a = \sigma^2 e_i^0(f)e_i^0 + \rho^2 e_r^0(f)e_r^0 = \sigma^2\mathrm{grad}_{g_0}f + (\rho^2 - \sigma^2)\mathcal{V}\mathrm{grad}_{g_0}f\,.$$

□

Recall that the basis $\{e_a^0\}$ is chosen such that at the point x_0, we have $\mathcal{V}\nabla^0_{e_r^0}e_s^0 = 0$, $\forall r, s = 3, 4$.

Lemma 4.6. *At the point x_0,*

$$\nabla_{e_a}e_a = \sigma^2(2\mu_0 + \mathcal{H}\mathrm{grad}\,g_0\ln(\sigma\lambda_0\rho^2)) + \rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho\sigma^2\lambda_0^2)$$

Proof. From Corollary 2.5(iv),

$$\nabla_{e_a}e_a = \mathrm{grad}\,\ln\lambda + \mathcal{V}\mathrm{grad}\,\ln\lambda + 2\mu + \omega_{34}(e_r)Je_r\,.$$

From Lemma 4.1, $\omega_{34}(e_r)Je_r = \mathcal{V}\mathrm{grad}\,\ln\rho$. The formula now follows from Lemmas 4.4 and 4.5. □

Define the vertical Laplacian at a point x with respect to the metric g of a smooth function f by $\Delta_g^V f = \Delta_g^F(f|_F) = e_r(e_r(f)) - \mathrm{d}f(\mathcal{V}\nabla_{e_r}e_r)$, where $F = \varphi^{-1}\varphi(x)$ is the fibre passing through x. Similarly, we have the vertical Laplacian with respect to g_0. Note that at the point x_0, we have $\Delta_{g_0}^V f = e_r^0(e_r^0(f))$.

Lemma 4.7. [3, 4]

$$\begin{aligned}\Delta_g f = \sigma^2\Delta_{g_0}f &+ (\rho^2 - \sigma^2)\{\Delta_{g_0}^V f - 2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0)\}\\ &- 2\sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}_{g_0}\ln\rho) - 2\rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\sigma)\,.\end{aligned}$$

Remark 4.8. Note that if $\sigma = \rho$, so that the transformation is conformal, we obtain the well-known formula for the transformation of the Laplacian:

$$\Delta_g f = \sigma^2\Delta_{g_0}f - (m-2)\sigma^2\mathrm{d}f(\mathrm{grad}_{g_0}\ln\sigma)$$

(with dimension $m = 4$).

Proof. From Lemma 4.6,

$$\begin{aligned}
\Delta_g f &= e_a(e_a(f)) - \mathrm{d}f(\nabla_{e_a} e_a) = e_i(e_i(f)) + e_r(e_r(f)) \\
&\quad - \mathrm{d}f\left(2\sigma^2\mu_0 + \sigma^2\mathcal{H}\mathrm{grad}\, g_0 \ln\sigma\lambda_0\rho^2 + \rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2\lambda_0^2\right) \\
&= \sigma^2 e_i^0(e_i^0(f)) + \sigma^2 e_i^0(\ln\sigma)e_i^0(f) + \rho^2 e_r^0(e_r^0(f)) + \rho^2 e_r^0(\ln\rho)e_r^0(f) \\
&\quad - \mathrm{d}f\left(2\sigma^2\mu_0 + \sigma^2\mathcal{H}\mathrm{grad}\, g_0 \ln(\sigma\lambda_0\rho^2) + \rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2\lambda_0^2\right) \\
&= \sigma^2\Delta_{g_0} f + (\rho^2-\sigma^2)e_r^0(e_r^0(f)) + \sigma^2\mathrm{d}f(\nabla^0_{e_a^0}e_a^0) + \sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}_{g_0}\ln\sigma) \\
&\quad + \rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\rho) \\
&\quad - 2\sigma^2\mathrm{d}f(\mu_0) - \sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}\, g_0\ln(\sigma\lambda_0\rho^2)) - \rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2\lambda_0^2)\,.
\end{aligned}$$

But from Corollary 2.5(iv), $\nabla^0_{e_a^0}e_a^0 = \mathrm{grad}_{g_0}\ln\lambda_0 + \mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0 + 2\mu_0$, so that

$$\begin{aligned}
\Delta_g f &= \sigma^2\Delta_{g_0} f + (\rho^2-\sigma^2)e_r^0(e_r^0(f)) + \sigma^2\mathrm{d}f(\mathrm{grad}_{g_0}\ln\lambda_0) + \sigma^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0) \\
&\quad + \sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}_{g_0}\ln\sigma) + \rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\rho) - \sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}\, g_0\ln(\sigma\lambda_0\rho^2)) \\
&\quad - \rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2\lambda_0^2) \\
&= \sigma^2\Delta_{g_0} f + (\rho^2-\sigma^2)\{\Delta^{\mathcal{V}}_{g_0} f - 2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0)\} \\
&\quad - 2\sigma^2\mathrm{d}f(\mathcal{H}\mathrm{grad}\, g_0\ln\rho) - 2\rho^2\mathrm{d}f(\mathcal{V}\mathrm{grad}_{g_0}\ln\sigma)\,.
\end{aligned}$$

□

Corollary 4.9.

$$\begin{aligned}
\Delta_g \ln\lambda &= \sigma^2\Delta_{g_0}\ln(\sigma\lambda_0) + (\rho^2-\sigma^2)\{\Delta^V_{g_0}(\ln(\sigma\lambda_0)) - 2\mathrm{d}\ln(\sigma\lambda_0)(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0)\} \\
&\quad - 2\sigma^2\mathrm{d}\ln(\sigma\lambda_0)(\mathcal{H}\mathrm{grad}_{g_0}\ln\rho) - 2\rho^2\mathrm{d}\ln(\sigma\lambda_0)(\mathcal{V}\mathrm{grad}_{g_0}\ln\sigma)\,.
\end{aligned}$$

4.2. **The second fundamental forms and their divergences.** The vertical components of the Ricci tensor contain the term $\operatorname{div} B_1$ acting on vertical vectors.

Lemma 4.10.

$$B_1(e_i, e_r, e_s) = B_1^0(e_i, e_r^0, e_s^0) + \delta_{rs}e_i(\ln\rho)\,.$$

Proof. This follows from Corollary 4.3:

$$\begin{aligned}
B_1(e_i, e_r, e_s) &= g(e_i, B_{e_r}e_s) = \frac{1}{\sigma^2}g_0(e_i, B_{e_r}e_s) \\
&= g_0(e_i, B^0_{e_r^0}e_s^0 + g_0(e_r^0, e_s^0)\mathcal{H}\mathrm{grad}_{g_0}\ln\rho) \\
&= B_1^0(e_i, e_r^0, e_s^0) + \delta_{rs}e_i(\ln\rho)\,.
\end{aligned}$$

□

Lemma 4.11.

$$\begin{aligned}
(\operatorname{div} B_1)(e_r, e_s) &= \sigma^2(\operatorname{div}_0 B_1^0)(e_r^0, e_s^0) - \sigma^2 B_1^0(\mathcal{H}\mathrm{grad}\, g_0\ln\rho^2, e_r^0, e_s^0) \\
&\quad + \delta_{rs}\sigma^2\big\{\mathrm{Tr}^H_{g_0}\nabla^0\mathrm{d}\ln\rho + 2\mathrm{d}\ln\rho(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0) \\
&\quad - \mathrm{d}\ln\rho(2\mu_0 + \mathcal{H}\mathrm{grad}\, g_0\ln\rho^2)\big\}\,.
\end{aligned}$$

(Note that $\mathrm{Tr}^H_{g_0}\nabla\mathrm{d}\ln\rho$ can be written in terms of the Laplacian and the vertical Laplacian).

Proof. Applying Lemma 4.10 and Lemma 4.6,

$$\begin{aligned}
(\operatorname{div} B_1)(e_r, e_s) &= (\nabla_{e_a} B_1)(e_a, e_r, e_s) = e_i(B_1(e_i, e_r, e_s)) - B_1(\nabla_{e_a} e_a, e_r, e_s) \\
&\quad - B_1(e_i, \nabla_{e_i} e_r, e_s) - B_1(e_i, e_r, \nabla_{e_i} e_s) \\
&= \sigma e_i^0(\sigma B_1^0(e_i^0, e_r^0, e_s^0) + \sigma\delta_{rs} e_i^0(\ln\rho)) \\
&\quad - \sigma^2 B_1(2\mu_0 + \mathcal{H}\operatorname{grad}_{g_0} \ln(\sigma\lambda_0\rho^2), e_r, e_s) \\
&\quad - B_1(e_i, \nabla_{e_i} e_r, e_s) - B_1(e_i, e_r, \nabla_{e_i} e_s) \\
&= \sigma^2 e_i^0(B_1^0(e_i^0, e_r^0, e_s^0)) + \sigma^2 e_i^0(\ln\sigma) B_1^0(e_i^0, e_r^0, e_s^0) \\
&\quad + \delta_{rs}\sigma^2 e_i^0(e_i^0(\ln\rho)) + \delta_{rs}\sigma^2 e_i^0(\ln\sigma) e_i^0(\ln\rho) \\
&\quad - \sigma^2 B_1(2\mu_0, \mathcal{H}\operatorname{grad}_{g_0} \ln(\sigma\lambda_0\rho^2), e_r, e_s) \\
&\quad - B_1(e_i, g_0(\nabla^0_{e_i} e_r^0, e_t^0) e_t, e_s) - B_1(e_i, e_r, g_0(\nabla^0_{e_i} e_s^0, e_t^0) e_t) \\
&= \sigma^2 \big\{ \operatorname{div}_0 B_1^0(e_r^0, e_s^0) + B_1^0(\nabla^0_{e_a^0} e_a^0, e_r^0, e_s^0) + B_1^0(e_i^0, \nabla^0_{e_i^0} e_r^0, e_s^0) \\
&\quad + B_1^0(e_i^0, e_r^0, \nabla^0_{e_i^0} e_s^0) + B_1^0(\mathcal{H}\operatorname{grad}_{g_0} \ln\sigma, e_r^0, e_s^0) \\
&\quad + \delta_{rs} e_i^0(e_i^0(\ln\rho)) + \delta_{rs} e_i^0(\ln\sigma) e_i^0(\ln\rho) \\
&\quad - B_1^0(2\mu_0 + \mathcal{H}\operatorname{grad}_{g_0} \ln(\sigma\lambda_0\rho^2), e_r^0, e_s^0) \\
&\quad - \delta_{rs}(2\mu_0 + \mathcal{H}\operatorname{grad}_{g_0} \ln(\sigma\lambda_0\rho^2))(\ln\rho) \\
&\quad - B_1^0(e_i^0, \nabla^0_{e_i^0} e_r^0, e_s^0) - B_1^0(e_i^0, e_r^0, \nabla^0_{e_i^0} e_s^0) - \big(g_0(\nabla^0_{e_i^0} e_s^0, e_r^0) \\
&\quad + g_0(\nabla^0_{e_i^0} e_r^0, e_s^0)\big) e_i(\ln\rho) \big\}\,.
\end{aligned}$$

After simplifying and noting that from Corollary 2.5(iii)

$$\begin{aligned}
\operatorname{Tr}^H_{g_0} \nabla^0 \mathrm{d}\ln\rho &= -\mathrm{d}\ln\rho(\nabla^0_{e_i^0} e_i^0) + e_i^0(e_i^0(\ln\rho)) \\
&= -\mathrm{d}\ln\rho\big(\operatorname{grad}_{g_0} \ln\lambda_0 + \mathcal{V}\operatorname{grad}_{g_0} \ln\lambda_0\big) + e_i^0(e_i^0(\ln\rho))\,,
\end{aligned}$$

the formula follows. □

Let us now deal with $\operatorname{div} B_2$. As for Lemma 4.10, we have

Lemma 4.12.

$$B_2(e_r, e_s, e_i) = B_2^0(e_r^0, e_s^0, e_i) + \delta_{rs} e_i(\ln\rho)\,.$$

Lemma 4.13.

$$\begin{aligned}
(\operatorname{div} B_2)(e_s, e_j) &= \operatorname{div}_{g_0} B_2^0(e_s, e_j) + \nabla^0 \mathrm{d}\ln\rho(e_s, e_j) - B_2^0(\mathcal{V}\operatorname{grad}_{g_0} \ln(\rho^3\sigma), e_s, e_j) \\
&\quad - e_s(\ln(\sigma{\lambda_0}^2)) e_j(\ln\rho) + 2 e_s(\ln\rho)\mu_0^\flat(e_j) \\
&\quad + \Big(\frac{\sigma^2}{\rho^2} - 1\Big) B_2^0(\zeta_0^\sharp, e_s, J e_j) + \Big(\frac{\sigma^2}{\rho^2} - 1\Big)\zeta_0(e_s) J e_j(\ln\rho)\,.
\end{aligned}$$

Proof.

$$\begin{aligned}
(\operatorname{div} B_2)(e_s, e_j) &= (\nabla_{e_a} B_2)(e_a, e_s, e_j) = e_r(B_2(e_r, e_s, e_j)) \\
&\quad - B_2(\mathcal{V}\nabla_{e_a} e_a, e_s, e_j) - B_2(e_r, \mathcal{V}\nabla_{e_r} e_s, e_j) - B_2(e_r, e_s, \mathcal{H}\nabla_{e_r} e_j)\,.
\end{aligned}$$

From Lemma 4.6, $\mathcal{V}\nabla_{e_a}e_a = \rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2\lambda_0^2$. From Lemma 4.1(iv)

$$\mathcal{V}\nabla_{e_r}e_s = g(\nabla_{e_r}e_s, e_t)e_t = (e_t(\ln\rho)\delta_{rs} - e_s(\ln\rho)\delta_{rt})e_t = \delta_{rs}\mathcal{V}\mathrm{grad}\,\ln\rho - e_s(\ln\rho)e_r,$$

and from Lemma 2.4(iii), $\mathcal{H}\nabla_{e_r}e_j = -\zeta(e_r)Je_j$. Thus, from Lemma 4.12,

$$\begin{aligned}
(\mathrm{div}\,B_2)(e_s,e_j) &= e_r(B_2(e_r,e_s,e_j)) - B_2(\rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho\sigma^2{\lambda_0}^2, e_s, e_j)\\
&\quad - B_2(e_r, \delta_{rs}\mathcal{V}\mathrm{grad}\,\ln\rho - e_s(\ln\rho)e_r, e_j) + \zeta(e_r)B_2(e_r,e_s,Je_j)\\
&= e_r(B_2(e_r,e_s,e_j)) - B_2(\rho^2\mathcal{V}\mathrm{grad}_{g_0}\ln\rho^2\sigma^2{\lambda_0}^2, e_s, e_j)\\
&\quad + 2\mu^\flat(e_j)e_s(\ln\rho) + \zeta(e_r)B_2(e_r,e_s,Je_j) = \rho e_r^0(B_2^0(e_r^0,e_s^0,e_j))\\
&\quad + e_s(e_j(\ln\rho)) - \rho B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma^2{\lambda_0}^2), e_s^0, e_j)\\
&\quad - e_s(\ln(\rho^2\sigma^2{\lambda_0}^2))e_j(\ln\rho) + 2e_s(\ln\rho)(\mu_0^\flat(e_j) + e_j(\ln\rho))\\
&\quad + \frac{\sigma^2}{\rho^2}\zeta_0(e_r)(B_2^0(e_r^0,e_s^0,Jej) + \delta_{rs}Je_j(\ln\rho)) = \rho e_r^0(B_2^0(e_r^0,e_s^0,e_j))\\
&\quad + e_s(e_j(\ln\rho)) - \rho B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma^2{\lambda_0}^2), e_s^0, e_j) - e_s(\ln(\sigma^2{\lambda_0}^2))e_j(\ln\rho)\\
&\quad + 2e_s(\ln\rho)\mu_0^\flat(e_j) + \frac{\sigma^2}{\rho^2}B_2^0(\zeta_0^\sharp, e_s, Jej) + \frac{\sigma^2}{\rho^2}\zeta_0(e_s)Je_j(\ln\rho))\\
&= \mathrm{div}_{g_0}B_2^0(e_s,e_j) + B_2^0(\mathcal{V}\nabla^0_{e_a^0}e_a^0, e_s, e_j) + B_2^0(e_r^0, e_s, \mathcal{H}\nabla^0_{e_r^0}e_j)\\
&\quad + e_s(e_j(\ln\rho)) - \rho B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma^2{\lambda_0}^2), e_s^0, e_j)\\
&\quad - e_s(\ln(\sigma^2{\lambda_0}^2))e_j(\ln\rho) + 2e_s(\ln\rho)\mu_0^\flat(e_j) + \frac{\sigma^2}{\rho^2}B_2^0(\zeta_0^\sharp, e_s, Jej)\\
&\quad + \frac{\sigma^2}{\rho^2}\zeta_0(e_s)Je_j(\ln\rho)) = \mathrm{div}_{g_0}B_2^0(e_s,e_j) + B_2^0(\mathcal{V}\nabla^0_{e_a^0}e_a^0, e_s, e_j)\\
&\quad + B_2^0(e_r^0, e_s, -\zeta_0(e_r^0)Je_j + e_r^0(\ln\sigma)e_j) + e_s(e_j(\ln\rho))\\
&\quad - B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma^2{\lambda_0}^2), e_s, e_j) - e_s(\ln(\sigma^2{\lambda_0}^2))e_j(\ln\rho)\\
&\quad + 2e_s(\ln\rho)\mu_0^\flat(e_j) + \frac{\sigma^2}{\rho^2}B_2^0(\zeta_0^\sharp, e_s, Jej) + \frac{\sigma^2}{\rho^2}\zeta_0(e_s)Je_j(\ln\rho))\\
&= \mathrm{div}_{g_0}B_2^0(e_s,e_j) + 2B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln\lambda_0, e_s, e_j) - B_2^0(\zeta_0^\sharp, e_s, Je_j)\\
&\quad + B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln\sigma, e_s, e_j) + e_s(e_j(\ln\rho))\\
&\quad - B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma^2{\lambda_0}^2), e_s, e_j) - e_s(\ln(\sigma^2{\lambda_0}^2))e_j(\ln\rho)\\
&\quad + 2e_s(\ln\rho)\mu_0^\flat(e_j) + \frac{\sigma^2}{\rho^2}B_2^0(\zeta_0^\sharp, e_s, Jej) + \frac{\sigma^2}{\rho^2}\zeta_0(e_s)Je_j(\ln\rho))\\
&= \mathrm{div}_{g_0}B_2^0(e_s,e_j) + e_s(e_j(\ln\rho)) - B_2^0(\mathcal{V}\mathrm{grad}_{g_0}\ln(\rho^2\sigma), e_s, e_j)\\
&\quad - e_s(\ln(\sigma^2{\lambda_0}^2))e_j(\ln\rho) + 2e_s(\ln\rho)\mu_0^\flat(e_j)\\
&\quad + \Big(\frac{\sigma^2}{\rho^2} - 1\Big)B_2^0(\zeta_0^\sharp, e_s, Jej) + \frac{\sigma^2}{\rho^2}\zeta_0(e_s)Je_j(\ln\rho)\,.
\end{aligned}$$

However

$$\begin{aligned}
\nabla^0 \mathrm{d} \ln \rho(e_s, e_j) &= -(\nabla^0_{e_s} e_j)(\ln \rho) + e_s(e_j(\ln \rho)) \\
&= -(\mathcal{H}\nabla^0_{e_s} e_j)(\ln \rho) - (\mathcal{V}\nabla^0_{e_s} e_j)(\ln \rho) + e_s(e_j(\ln \rho) \\
&= \zeta_0(e_s) J e_j(\ln \rho) - e_s(\ln \sigma) e_j(\ln \rho) + B^{0*}_{e_s} e_j(\ln \rho) + e_s(e_j(\ln \rho))\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
B^{0*}_{e_s} e_j(\ln \rho) &= g_0(\mathcal{V}\mathrm{grad}\ \ln \rho, B^{0*}_{e_s} e_j) = g_0(B^0_{e_s} \mathcal{V}\mathrm{grad}\ \ln \rho, e_j) \\
&= B^0_2(\mathcal{V}\mathrm{grad}\ \ln \rho, e_s, e_j)
\end{aligned}$$

and the expression follows. □

Lemma 4.14. *Under biconformal deformation, the quantities C and C^* change according to*

$$\begin{aligned}
C &= \frac{\sigma^2}{\rho^2}\left\{C_0 + \mathrm{d}\ln\rho(B^0_\star \star) + ||\mathcal{H}\mathrm{grad}_{\,g_0} \ln \rho||^2_{g_0} g_0^{\mathcal{V}}\right\} \\
C^* &= C^*_0 + 4\mathrm{d}\ln\rho \odot \mu^\flat_0 + 2(\mathrm{d}\ln\rho \circ \mathcal{H})^2\,.
\end{aligned}$$

Proof. From Corollary (4.3),

$$\begin{aligned}
C(e_r, e_s) &= g(B_{e_t} e_r, B_{e_t} e_s) = \frac{1}{\sigma^2} g_0(B_{e_t} e_r, B_{e_t} e_s) \\
&= \frac{1}{\sigma^2} g_0(\sigma^2(B^0_{e^0_t} e^0_r + \delta_{rt}\mathcal{H}\mathrm{grad}_{\,g_0} \ln \rho), \sigma^2(B^0_{e^0_t} e^0_s + \delta_{st}\mathcal{H}\mathrm{grad}_{\,g_0} \ln \rho)) \\
&= \frac{\sigma^2}{\rho^2} C_0(e_r, e_s) + \frac{2\sigma^2}{\rho^2}\mathrm{d}\ln\rho(B^0_{e_r} e_s) + \frac{\sigma^2}{\rho^2}||\mathcal{H}\mathrm{grad}_{\,g_0} \ln\rho||^2_{g_0} g_0(e_r, e_s)\,,
\end{aligned}$$

whereas

$$\begin{aligned}
C^*(e_i, e_j) &= g(B^*_{e_r} e_i, B^*_{e_r} e_j) = g(e_s, B^*_{e_r} e_i) g(e_s, B^*_{e_r} e_j) = g(B_{e_r} e_s, e_i) g(B_{e_r} e_s, e_j) \\
&= \frac{1}{\sigma^4} g_0(\sigma^2(B^0_{e^0_r} e^0_s + \delta_{rs}\mathcal{H}\mathrm{grad}_{\,g_0} \ln\rho), e_i) \\
&\quad \times g_0(\sigma^2(B^0_{e^0_r} e^0_s + \delta_{rs}\mathcal{H}\mathrm{grad}_{\,g_0} \ln\rho), e_j) \\
&= C^*_0(e_i, e_j) + 2\mathrm{d}\ln\rho(e_i)\mu^\flat_0(e_j) + 2\mathrm{d}\ln\rho(e_j)\mu^\flat_0(e_i) + 2\mathrm{d}\ln\rho(e_i)\mathrm{d}\ln\rho(e_j)\,.
\end{aligned}$$

□

Remark 4.15. When $\sigma = \rho$, the deformation is conformal and there is a well-known formula for the change in Ricci [7]:

$$\begin{aligned}
\mathrm{Ric}\,(e_a, e_b) &= \mathrm{Ric}^{\,0}(e_a, e_b) + 2[\nabla^0 \mathrm{d}\ln\sigma(e_a, e_b) + e_a(\ln\sigma) e_b(\ln\sigma)] \\
&\quad + (\Delta_{g_0} \ln\sigma - 2\|\mathrm{grad}_{\,g_0} \ln\sigma\|^2) g_0(e_a, e_b)\,.
\end{aligned}$$

5. Orthogonal projection from $\mathbb{R}^4$ to $\mathbb{R}^2$

Let $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the canonical projection $\varphi(x^1, x^2, x^3, x^4) = (x^1, x^2)$. Then $\lambda_0 \equiv 1, \mu_0 \equiv 0, B^0 = B^0_1 = B^0_2 \equiv 0, \zeta^0 \equiv 0$. We take the standard basis: $e^0_a = \partial/\partial x^a$.

From Lemma 3.4,

$$\begin{aligned}\mathrm{Ric}\,|_{H\times H} &= \left\{\lambda^2 K^N + \Delta\ln\lambda + 2\mathrm{d}\ln\lambda(\mu) - 2\|\zeta\|^2\right\} g^{\mathcal{H}} - C^* + \mathcal{L}_\mu g|_{H\times H}\\ &= \{\Delta\ln\lambda + 2\mathrm{d}\ln\lambda(\mu)\}\, g^{\mathcal{H}} - C^* + \mathcal{L}_\mu g|_{H\times H}\,,\end{aligned}$$

where $\lambda = \sigma$ and $\mu = \sigma^2\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho$.

From Corollary 4.9,

$$\begin{aligned}\Delta_g\ln\lambda = {}&\sigma^2\Delta_{g_0}\ln\sigma + (\rho^2-\sigma^2)\Delta^{\mathcal{V}}_{g_0}(\ln\sigma) - 2\sigma^2\mathrm{d}\ln\sigma(\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho)\\ &- 2\rho^2\mathrm{d}\ln\sigma(\mathcal{V}\mathrm{grad}_{\,g_0}\ln\sigma)\end{aligned}$$

and $\mathrm{d}\ln\lambda(\mu) = \sigma^2\mathrm{d}\ln\sigma(\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho)$, so that

$$\begin{aligned}\Delta_g\ln\lambda + 2\mathrm{d}\ln\lambda(\mu) = {}&\sigma^2\left(\frac{\partial^2\ln\sigma}{\partial {x_1}^2} + \frac{\partial^2\ln\sigma}{\partial {x_2}^2}\right) + \rho^2\left(\frac{\partial^2\ln\sigma}{\partial {x_3}^2} + \frac{\partial^2\ln\sigma}{\partial {x_4}^2}\right)\\ &- 2\rho^2\left(\left(\frac{\partial\ln\sigma}{\partial x_3}\right)^2 + \left(\frac{\partial\ln\sigma}{\partial x_4}\right)^2\right).\end{aligned}$$

From Lemma 4.14,

$$C^*(e_i, e_j) = 2\sigma^2 e^0_i(\ln\rho)e^0_j(\ln\rho) = 2\sigma^2\frac{\partial\ln\rho}{\partial x^i}\frac{\partial\ln\rho}{\partial x^j}\,.$$

Next, from Lemma 4.1(v),

$$\begin{aligned}\mathcal{L}_\mu g(e_i, e_j) &= g(\nabla_{e_i}\mu, e_j) + g(e_i, \nabla_{e_j}\mu)\\ &= e_i(g(\mu, e_j)) + e_j(g(\mu, e_i)) - g(\mu, \nabla_{e_i}e_j + \nabla_{e_j}e_i)\\ &= e_i(e_j(\ln\rho)) + e_j(e_i(\ln\rho)) - e_k(\ln\rho)g(e_k, \nabla_{e_i}e_j + \nabla_{e_j}e_i)\\ &= e_i(e_j(\ln\rho)) + e_j(e_i(\ln\rho)) + e_i(\ln\rho)e_j(\ln\sigma)\\ &\quad + e_i(\ln\sigma)e_j(\ln\rho) - 2\delta_{ij}\mathcal{H}\mathrm{grad}_{\,g}\ln\rho\\ &= 2\sigma^2\left\{\frac{\partial^2\ln\rho}{\partial x^i\partial x^j} + \frac{\partial\ln\sigma}{\partial x^i}\frac{\partial\ln\rho}{\partial x^j} + \frac{\partial\ln\rho}{\partial x^i}\frac{\partial\ln\sigma}{\partial x^j}\right.\\ &\quad \left. - \delta_{ij}\left(\frac{\partial\ln\sigma}{\partial x_1}\frac{\partial\ln\rho}{\partial x_1} + \frac{\partial\ln\sigma}{\partial x_2}\frac{\partial\ln\rho}{\partial x_2}\right)\right\}.\end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned}\mathrm{Ric}\,(e_1,e_1) = \sigma^2\Big\{ & \left(\frac{\partial^2\ln\sigma}{\partial {x_1}^2}+\frac{\partial^2\ln\sigma}{\partial {x_2}^2}\right)+\frac{\rho^2}{\sigma^2}\left(\frac{\partial^2\ln\sigma}{\partial {x_3}^2}+\frac{\partial^2\ln\sigma}{\partial {x_4}^2}\right)\\ & -2\frac{\rho^2}{\sigma^2}\left(\left(\frac{\partial\ln\sigma}{\partial x_3}\right)^2+\left(\frac{\partial\ln\sigma}{\partial x_4}\right)^2\right)\\ & -2\left(\frac{\partial\ln\rho}{\partial x_1}\right)^2+2\frac{\partial^2\ln\rho}{\partial x_1^2}+2\frac{\partial\ln\sigma}{\partial x_1}\frac{\partial\ln\rho}{\partial x_1}-2\frac{\partial\ln\sigma}{\partial x_2}\frac{\partial\ln\rho}{\partial x_2}\Big\}\\ \mathrm{Ric}\,(e_2,e_2) = \sigma^2\Big\{ & \left(\frac{\partial^2\ln\sigma}{\partial {x_1}^2}+\frac{\partial^2\ln\sigma}{\partial {x_2}^2}\right)+\frac{\rho^2}{\sigma^2}\left(\frac{\partial^2\ln\sigma}{\partial {x_3}^2}+\frac{\partial^2\ln\sigma}{\partial {x_4}^2}\right)\\ & -2\frac{\rho^2}{\sigma^2}\left(\left(\frac{\partial\ln\sigma}{\partial x_3}\right)^2+\left(\frac{\partial\ln\sigma}{\partial x_4}\right)^2\right)\\ & -2\left(\frac{\partial\ln\rho}{\partial x_2}\right)^2+2\frac{\partial^2\ln\rho}{\partial x_2^2}+2\frac{\partial\ln\sigma}{\partial x_2}\frac{\partial\ln\rho}{\partial x_2}-2\frac{\partial\ln\sigma}{\partial x_1}\frac{\partial\ln\rho}{\partial x_1}\Big\}\\ \mathrm{Ric}\,(e_1,e_2) = 2\sigma^2\Big\{ & \frac{\partial^2\ln\rho}{\partial x_1\partial x_2}-\frac{\partial\ln\rho}{\partial x_1}\frac{\partial\ln\rho}{\partial x_2}+\frac{\partial\ln\sigma}{\partial x_1}\frac{\partial\ln\rho}{\partial x_2}+\frac{\partial\ln\rho}{\partial x_1}\frac{\partial\ln\sigma}{\partial x_2}\Big\}\,.\end{aligned}$$

From Lemma 3.5, the mixed Ricci tensor acting on (e_j,e_s) is given by

$$\begin{aligned}\mathrm{Ric}\,(e_j,e_s) &= \nabla\mathrm{d}\ln\lambda(e_j,e_s)-(\mathrm{d}\ln\lambda)^2(e_j,e_s)-2(\mathrm{d}\ln\lambda\odot\zeta)(Je_j,e_s)\\ &\quad-(\nabla_{Je_j}\zeta)(e_s)-2\zeta(\nabla_{e_s}Je_j)\\ &\quad-\mathrm{div}\,B_2(e_s,e_j)-2\mathrm{d}\ln\lambda(B^*_{e_s}e_j)+2(\nabla_{e_s}\mu^\flat)(e_j)\\ &=\nabla\mathrm{d}\ln\sigma(e_j,e_s)-(\mathrm{d}\ln\sigma)^2(e_j,e_s)-\mathrm{div}\,B_2(e_s,e_j)\\ &\quad-2\mathrm{d}\ln\lambda(B^*_{e_s}e_j)+2(\nabla_{e_s}\mu^\flat)(e_j)\,.\end{aligned}$$

From Corollary 4.2,

$$\nabla\mathrm{d}\ln\sigma(e_s,e_j)=e_s(e_j(\ln\sigma))-\mathrm{d}\ln\sigma(\nabla_{e_s}e_j)=e_s(e_j(\ln\sigma))+e_j(\ln\rho)e_s(\ln\sigma)\,.$$

Since the fibres before deformation are totally geodesic, $B^0\equiv 0$, so from Lemma 4.13,

$$\begin{aligned}(\mathrm{div}\,B_2)(e_s,e_j) &= \nabla^0\mathrm{d}\ln\rho(e_s,e_j)-e_s(\ln\sigma)e_j(\ln\rho)=e_s(e_j(\ln\rho))\\ &\quad-\mathrm{d}\ln\rho(\nabla^0_{e_s}e_j)-e_s(\ln\sigma)e_j(\ln\rho)\\ &=e_s(e_j(\ln\rho))-\mathrm{d}\ln\rho\big(\sigma\rho\nabla^0_{e_s^0}e_j^0+\sigma\rho e_s^0(\ln\sigma)e_j^0\big)-e_s(\ln\sigma)e_j(\ln\rho)\\ &=e_s(e_j(\ln\rho))-2e_s(\ln\sigma)e_j(\ln\rho)\,.\end{aligned}$$

From Corollary (4.2),

$$\mathrm{d}\ln\lambda(B^*_{e_s}e_j)=-\mathrm{d}\ln\sigma(\mathcal{V}\nabla_{e_s}e_j)=-\mathrm{d}\ln\sigma(e_r)g(e_r,\nabla_{e_s}e_j)=e_s(\ln\sigma)e_j(\ln\rho)\,.$$

Finally, $\mu=\sigma^2\mathcal{H}\mathrm{grad}_{g_0}\ln\rho=e_i(\ln\rho)e_i$, so that from Corollary 4.2,

$$\begin{aligned}(\nabla_{e_s}\mu^\flat)(e_j) &= e_s(g(\mu,e_j))-g(\mu,\nabla_{e_s}e_j)\\ &=e_s(e_i(\ln\rho)\delta_{ij})-e_i(\ln\rho)g(e_i,\nabla_{e_s}e_j)=e_s(e_j(\ln\rho))\,.\end{aligned}$$

We conclude that

$$\mathrm{Ric}\,(e_j, e_s) = e_s(e_j(\ln\sigma)) + e_s(e_j(\ln\rho)) + e_j(\ln\rho)e_s(\ln\sigma) - e_j(\ln\sigma)e_s(\ln\sigma),$$

explicitly

$$\mathrm{Ric}\,(e_j, e_s) = \sigma\rho\left\{\frac{\partial^2\ln(\sigma\rho)}{\partial x^j\partial x^s} + 2\frac{\partial\ln\sigma}{\partial x^s}\frac{\partial\ln\rho}{\partial x^j}\right\}.$$

The vertical components of the Ricci tensor are given by

$$\begin{aligned}\mathrm{Ric}\,|_{V\times V} &= K^V g^V + 2\nabla\mathrm{d}\ln\lambda|_{V\times V} + 2\mathrm{d}\ln\lambda(B_\star\star) - 2(\mathrm{d}\ln\lambda)^2|_{V\times V}\\ &\quad + 2\zeta^2 + \mathrm{div}\,B_1|_{V\times V}\\ &= K^V g^V + 2\nabla\mathrm{d}\ln\lambda|_{V\times V} + 2\mathrm{d}\ln\lambda(B_\star\star) - 2(\mathrm{d}\ln\lambda)^2|_{V\times V} + \mathrm{div}\,B_1|_{V\times V}\,.\end{aligned}$$

After biconformal deformation, the sectional curvature of the fibres is given by

$$K^V = \rho^2\Delta^V_{g_0}\ln\rho\,.$$

For the second fundamental form:

$$\nabla\mathrm{d}\ln\lambda(e_r, e_s) = e_r(e_s(\ln\lambda)) - \mathrm{d}\ln\lambda(\nabla_{e_r}e_s)\,.$$

From Corollary 4.2,

$$\nabla_{e_r}e_s = \delta_{rs}\{\sigma^2\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho + \rho^2\mathcal{V}\mathrm{grad}_{\,g_0}\ln\rho\} - e_s(\ln\rho)e_r$$

and

$$\begin{aligned}\nabla\mathrm{d}\ln\lambda(e_r, e_s) &= e_r(e_s(\ln\lambda)) - \mathrm{d}\ln\lambda(\nabla_{e_r}e_s)\\ &= \rho^2 e^0_r(e^0_s(\ln\sigma)) + \rho^2 e^0_r(\ln\rho)e^0_s(\ln\sigma) + \rho^2 e^0_s(\ln\rho)e^0_r(\ln\sigma)\\ &\quad - \delta_{rs}\mathrm{d}\ln\sigma(\sigma^2\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho + \rho^2\mathcal{V}\mathrm{grad}_{\,g_0}\ln\rho)\end{aligned}$$

From Corollary (4.3),

$$B_{e_r}e_s = \sigma^2\delta_{rs}\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho\,.$$

From Lemma 4.11 we have

$$(\mathrm{div}\,B_1)(e_r, e_s) = \delta_{rs}\sigma^2\{\mathrm{Tr}^{\,\mathcal{H}}_{g_0}\nabla\mathrm{d}\ln\rho - \mathrm{d}\ln\rho(\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho^2)\}\,.$$

Thus

$$\begin{aligned}\mathrm{Ric}\,(e_r, e_s) &= \rho^2\delta_{rs}\Delta^V_{g_0}\ln\rho - 2\rho^2 e^0_r(\ln\sigma)e^0_s(\ln\sigma)\\ &\quad + 2\{\rho^2 e^0_r(e^0_s(\ln\sigma)) + \rho^2 e^0_r(\ln\rho)e^0_s(\ln\sigma) + \rho^2 e^0_s(\ln\rho)e^0_r(\ln\sigma)\}\\ &\quad + \delta_{rs}\{\sigma^2\mathrm{Tr}^{\,\mathcal{H}}_{g_0}\nabla\mathrm{d}\ln\rho - 2\sigma^2\mathrm{d}\ln\rho(\mathcal{H}\mathrm{grad}_{\,g_0}\ln\rho)\\ &\quad - 2\rho^2\mathrm{d}\ln\sigma(\mathcal{V}\mathrm{grad}_{\,g_0}\ln\rho)\}\,.\end{aligned}$$

Explicitly,

$$
\begin{aligned}
\mathrm{Ric}\,(e_r,e_s) = \rho^2\Bigg\{ & 2\frac{\partial^2 \ln\sigma}{\partial x^r\partial x^s} + 2\frac{\partial\ln\rho}{\partial x^r}\frac{\partial\ln\sigma}{\partial x^s} + 2\frac{\partial\ln\sigma}{\partial x^r}\frac{\partial\ln\rho}{\partial x^s} - 2\frac{\partial\ln\sigma}{\partial x^r}\frac{\partial\ln\sigma}{\partial x^s}\\
& + \delta_{rs}\Bigg(\frac{\sigma^2}{\rho^2}\left(\frac{\partial^2\ln\rho}{\partial {x_1}^2} + \frac{\partial^2\ln\rho}{\partial {x_2}^2}\right) + \frac{\partial^2\ln\rho}{\partial {x_3}^2} + \frac{\partial^2\ln\rho}{\partial {x_4}^2}\\
& - 2\frac{\sigma^2}{\rho^2}\left(\left(\frac{\partial\ln\rho}{\partial x_1}\right)^2 + \left(\frac{\partial\ln\rho}{\partial x_2}\right)^2\right) - 2\left(\frac{\partial\ln\sigma}{\partial x_3}\frac{\partial\ln\rho}{\partial x_3} + \frac{\partial\ln\sigma}{\partial x_4}\frac{\partial\ln\rho}{\partial x_4}\right)\Bigg)\Bigg\}.
\end{aligned}
$$

The equations for an Einstein metric: Ric $= Ag$ for some constant A, become the following system of ten equations:

(2)

$$
\begin{aligned}
\text{(i)}\quad A = \sigma^2\Bigg\{ & \frac{\partial^2\ln\sigma}{\partial {x_1}^2} + \frac{\partial^2\ln\sigma}{\partial {x_2}^2} + \frac{\rho^2}{\sigma^2}\left(\frac{\partial^2\ln\sigma}{\partial {x_3}^2} + \frac{\partial^2\ln\sigma}{\partial {x_4}^2}\right)\\
& - 2\frac{\rho^2}{\sigma^2}\left(\left(\frac{\partial\ln\sigma}{\partial x_3}\right)^2 + \left(\frac{\partial\ln\sigma}{\partial x_4}\right)^2\right) + 2\frac{\partial^2\ln\rho}{\partial x_j^2}\\
& - 2\left(\frac{\partial\ln\rho}{\partial x_j}\right)^2 + 2\frac{\partial\ln\sigma}{\partial x_j}\frac{\partial\ln\rho}{\partial x_j} - 2\frac{\partial\ln\sigma}{\partial x_{j'}}\frac{\partial\ln\rho}{\partial x_{j'}}\Bigg\} \quad (j=1,2)
\end{aligned}
$$

$$
\text{(ii)}\quad 0 = \frac{\partial^2\ln\rho}{\partial x_1\partial x_2} + \frac{\partial\ln\sigma}{\partial x_1}\frac{\partial\ln\rho}{\partial x_2} + \frac{\partial\ln\rho}{\partial x_1}\frac{\partial\ln\sigma}{\partial x_2} - \frac{\partial\ln\rho}{\partial x_1}\frac{\partial\ln\rho}{\partial x_2}
$$

$$
\text{(iii)}\quad 0 = \frac{\partial^2\ln(\sigma\rho)}{\partial x^j\partial x^s} + 2\frac{\partial\ln\sigma}{\partial x^s}\frac{\partial\ln\rho}{\partial x^j} \quad (j=1,2,\ s=3,4)
$$

$$
\begin{aligned}
\text{(iv)}\quad A = \rho^2\Bigg\{ & 2\frac{\partial^2\ln\sigma}{\partial {x_s}^2} - 2\left(\frac{\partial\ln\sigma}{\partial x_s}\right)^2 + 2\frac{\partial\ln\sigma}{\partial x_s}\frac{\partial\ln\rho}{\partial x_s} - 2\frac{\partial\ln\sigma}{\partial x_{s'}}\frac{\partial\ln\rho}{\partial x_{s'}} \quad (s=3,4)\\
& + \frac{\sigma^2}{\rho^2}\left(\frac{\partial^2\ln\rho}{\partial {x_1}^2} + \frac{\partial^2\ln\rho}{\partial {x_2}^2}\right) + \frac{\partial^2\ln\rho}{\partial {x_3}^2} + \frac{\partial^2\ln\rho}{\partial {x_4}^2}\\
& - 2\frac{\sigma^2}{\rho^2}\left(\left(\frac{\partial\ln\rho}{\partial x_1}\right)^2 + \left(\frac{\partial\ln\rho}{\partial x_2}\right)^2\right)\Bigg\}
\end{aligned}
$$

$$
\text{(v)}\quad 0 = \frac{\partial^2\ln\sigma}{\partial x_3\partial x_4} + \frac{\partial\ln\rho}{\partial x_3}\frac{\partial\ln\sigma}{\partial x_4} + \frac{\partial\ln\sigma}{\partial x_3}\frac{\partial\ln\rho}{\partial x_4} - \frac{\partial\ln\sigma}{\partial x_3}\frac{\partial\ln\sigma}{\partial x_4}\,.
$$

Note the symmetry between the equations: after the interchange $(j,k,\sigma,\rho)\leftrightarrow(r,s,\rho,\sigma)$, equations (i) and (iv) are interchanged.

5.1. **Warped product solutions.** Let us investigate some special solutions. If $\sigma=\sigma(x_1,x_2)$ and $\rho=\rho(x_3,x_4)$, then the system reduces to

$$
\frac{\partial^2\ln\sigma}{\partial {x_1}^2} + \frac{\partial^2\ln\sigma}{\partial {x_2}^2} = A/\sigma^2 \qquad \text{and} \qquad \frac{\partial^2\ln\rho}{\partial {x_3}^2} + \frac{\partial^2\ln\rho}{\partial {x_4}^2} = A/\rho^2\,.
$$

Note that $A = \sigma^2 \left(\frac{\partial^2 \ln \sigma}{\partial {x_1}^2} + \frac{\partial^2 \ln \sigma}{\partial {x_2}^2} \right)$ is the Gaussian curvature of the surface $(\mathbb{R}^2, (\mathrm{d}{x_1}^2 + \mathrm{d}{x_2}^2)/\sigma^2)$; similarly for the second equation. For example, setting

$$\sigma = \frac{1 + {x_1}^2 + {x_2}^2}{2} \qquad \text{and} \qquad \rho = \frac{1 + {x_3}^2 + {x_4}^2}{2}$$

yields the product of spheres $S^2 \times S^2$ with constant $A = 1$, whereas setting

$$\sigma = \frac{1 - {x_1}^2 - {x_2}^2}{2} \qquad \text{and} \qquad \rho = \frac{1 - {x_3}^2 - {x_4}^2}{2}$$

yields the product of hyperbolic spaces $H^2 \times H^2$ with constant $A = -1$.

More generally a warped product of the surfaces $(\mathbb{R}^2, (\mathrm{d}{x_1}^2 + \mathrm{d}{x_2}^2)/\sigma(x_1, x_2)^2)$ and $(\mathbb{R}^2, (\mathrm{d}{x_3}^2 + \mathrm{d}{x_4}^2)/\beta(x_3, x_4)^2)$ corresponds to $\mathbb{R}^4$ endowed with a metric of the form:

$$g = \frac{\mathrm{d}{x_1}^2 + \mathrm{d}{x_2}^2}{\sigma(x_1, x_2)^2} + \frac{\mathrm{d}{x_3}^2 + \mathrm{d}{x_4}^2}{\alpha(x_1, x_2)^2 \beta(x_3, x_4)^2}\,. \tag{3}$$

Setting $\sigma = \sigma(x_1, x_2)$ and $\rho = \alpha(x_1, x_2)\beta(x_3, x_4)$, the Einstein equations become the system:

$$\begin{aligned}
&\text{(i)} \quad A = \sigma^2 \Bigg\{ \frac{\partial^2 \ln \sigma}{\partial {x_1}^2} + \frac{\partial^2 \ln \sigma}{\partial {x_2}^2} - 2 \left(\frac{\partial \ln \alpha}{\partial x_1} \right)^2 + 2 \frac{\partial^2 \ln \alpha}{\partial x_1^2} \\
&\qquad\qquad + 2 \frac{\partial \ln \sigma}{\partial x_1} \frac{\partial \ln \alpha}{\partial x_1} - 2 \frac{\partial \ln \sigma}{\partial x_2} \frac{\partial \ln \alpha}{\partial x_2} \Bigg\} \\
&\text{(ii)} \quad A = \sigma^2 \Bigg\{ \frac{\partial^2 \ln \sigma}{\partial {x_1}^2} + \frac{\partial^2 \ln \sigma}{\partial {x_2}^2} - 2 \left(\frac{\partial \ln \alpha}{\partial x_2} \right)^2 + 2 \frac{\partial^2 \ln \alpha}{\partial x_2^2} \\
&\qquad\qquad + 2 \frac{\partial \ln \sigma}{\partial x_2} \frac{\partial \ln \alpha}{\partial x_2} - 2 \frac{\partial \ln \sigma}{\partial x_1} \frac{\partial \ln \alpha}{\partial x_1} \Bigg\} \\
&\text{(iii)} \quad 0 = \frac{\partial^2 \ln \alpha}{\partial x_1 \partial x_2} + \frac{\partial \ln \sigma}{\partial x_1} \frac{\partial \ln \alpha}{\partial x_2} + \frac{\partial \ln \alpha}{\partial x_1} \frac{\partial \ln \sigma}{\partial x_2} - \frac{\partial \ln \alpha}{\partial x_1} \frac{\partial \ln \alpha}{\partial x_2} \\
&\text{(iv)} \quad A = \sigma^2 \left(\frac{\partial^2 \ln \alpha}{\partial {x_1}^2} + \frac{\partial^2 \ln \alpha}{\partial {x_2}^2} \right) + \alpha^2 \beta^2 \left(\frac{\partial^2 \ln \beta}{\partial {x_3}^2} + \frac{\partial^2 \ln \beta}{\partial {x_4}^2} \right) \\
&\qquad\qquad - 2\sigma^2 \left(\left(\frac{\partial \ln \alpha}{\partial x_1} \right)^2 + \left(\frac{\partial \ln \alpha}{\partial x_2} \right)^2 \right) .
\end{aligned} \tag{4}$$

The sum of (i) and (ii) gives the equation:

$$A = \sigma^2 \left(\Delta_{g_0} \ln \sigma + \Delta_{g_0} \ln \alpha - ||\mathrm{grad}_{\,g_0} \ln \alpha||_0^2 \right) .$$

On the other hand the difference gives:

$$0 = \frac{\partial^2 \ln \alpha}{\partial {x_1}^2} - \frac{\partial^2 \ln \alpha}{\partial {x_2}^2} - \left(\frac{\partial \ln \alpha}{\partial x_1} \right)^2 + \left(\frac{\partial \ln \alpha}{\partial x_2} \right)^2 + 2 \frac{\partial \ln \sigma}{\partial x_1} \frac{\partial \ln \alpha}{\partial x_1} - 2 \frac{\partial \ln \sigma}{\partial x_2} \frac{\partial \ln \alpha}{\partial x_2} .$$

Since $\alpha = \alpha(x_1, x_2)$ and $\beta = \beta(x_3, x_4)$ are independent, on dividing equation (iv) by α^2, we deduce that

$$\beta^2 \left(\frac{\partial^2 \ln \beta}{\partial {x_3}^2} + \frac{\partial^2 \ln \beta}{\partial {x_4}^2} \right) = C \tag{5}$$

for a constant C and in particular the metric $(\mathrm{d}{x_3}^2 + \mathrm{d}{x_4}^2)/\beta^2$ is necessarily of constant Gaussian curvature C, and

$$A - C\alpha^2 = \sigma^2 \Delta_{g_0} \ln \alpha - 2\sigma^2 \|\mathrm{grad}_{\,g_0} \ln \alpha\|_0^2 \,.$$

Set $x_1 = t$ and suppose that $\alpha = \alpha(t)$ and $\sigma = \sigma(t)$ depend only on t. Then (4)(iii) is satisfied and α and σ are determined by the system:

$$\begin{cases} \text{(i)} \quad A = \sigma^2 \left((\ln \sigma)'' + (\ln \alpha)'' - (\ln \alpha)'^2 \right) \\ \text{(ii)} \quad 0 = (\ln \alpha)'' - (\ln \alpha)'^2 + 2(\ln \sigma)'(\ln \alpha)' \\ \text{(iii)} \quad A - C\alpha^2 = \sigma^2 \left((\ln \alpha)'' - 2(\ln \alpha)'^2 \right) \end{cases} . \tag{6}$$

From (6)(ii), provided $(\ln \alpha)' \not\equiv 0$,

$$\begin{aligned} 2(\ln \sigma)' &= \frac{-(\ln \alpha)'' + (\ln \alpha)'^2}{(\ln \alpha)'} = (-\ln |(\ln \alpha)'| + \ln \alpha)' \\ \implies \; 2 \ln \sigma &= -\ln |(\ln \alpha)'| + \ln \alpha + a \implies \sigma^2 = B\alpha^2/\alpha' \,, \end{aligned}$$

for constants a and B, with B non-zero. In particular, taking the difference between (6)(i) and (iii), we deduce that

$$\widetilde{C}\alpha' = (\ln \sigma)'' + (\ln \alpha)'^2 \,,$$

where $\widetilde{C} = C/B$. But from (6)(ii),

$$\begin{aligned} 2(\ln \sigma)'' &= \frac{-(\ln \alpha)''' + (\ln \alpha)'(\ln \alpha)''}{(\ln \alpha)'} + \frac{(\ln \alpha)''^2}{(\ln \alpha)'^2} \\ \implies \; 2\widetilde{C}\alpha' &= \frac{-(\ln \alpha)''' + (\ln \alpha)'(\ln \alpha)''}{(\ln \alpha)'} + \frac{(\ln \alpha)''^2}{(\ln \alpha)'^2} + 2(\ln \alpha)'^2. \end{aligned}$$

This simplifies to the third order ODE:

$$\alpha\alpha''' = 2\alpha'\alpha'' + \frac{\alpha(\alpha'')^2}{\alpha'} - 2\widetilde{C}\alpha\alpha'^2 \,.$$

Note the specific solution $\alpha(t) = t$ corresponding to hyperbolic space. More generally, if we set $\gamma(t) = \alpha'(t)$, $\delta(t) = \alpha''(t) = \gamma'(t)$, then we have the first order system:

$$\begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix}' = \begin{pmatrix} \gamma \\ \delta \\ \frac{2\gamma\delta}{\alpha} + \frac{\delta^2}{\gamma} - 2\widetilde{C}\gamma^2 \end{pmatrix} . \tag{7}$$

Cauchy's existence theorem (see, for example, [5] (10.4.5)) yields local solutions: Let $\Gamma_0 = \begin{pmatrix} \alpha_0 \\ \gamma_0 \\ \delta_0 \end{pmatrix} \in \mathbb{R}^3$ be a point with $\alpha_0 > 0$ and $\gamma_0 \neq 0$ and let $t_0 \in \mathbb{R}$. Then there is a solution $\Gamma(t) = \begin{pmatrix} \alpha(t) \\ \gamma(t) \\ \delta(t) \end{pmatrix}$ to (7) defined on an open interval $I \subset \mathbb{R}$ ($t_0 \in I$), with $\Gamma(t_0) = \Gamma_0$.

Given such a solution to (7) on an open interval I with $\alpha(t)$ positive and $\alpha'(t)$ non-zero for all $t \in I$, then defining σ by $\sigma^2 = B\alpha^2/\alpha'$, where B is a non-zero constant of sign consistent with α' and where we require $C = B\widetilde{C}$ to be the constant Gaussian curvature of the metric $(\mathrm{d}{x_3}^2 + \mathrm{d}{x_4}^2)/\beta(x_3, x_4)^2$, equations (6) are satisfied and the metric (3) is Einstein. The constant A is given by (6)(iii):

$$A = C\alpha^2 + \frac{B\alpha^2}{\alpha'}\left(\frac{\alpha''}{\alpha} - \frac{3\alpha'^2}{\alpha^2}\right),$$

which one easily checks is an integral of (7).

5.2. **Solutions depending on a single parameter.** Replace x_1 with the parameter t and suppose that both σ and ρ depend only on t. Then (2)(iii), (iv) and (vii) are satisfied, while (i) becomes

$$A = \sigma^2\big\{(\ln\sigma)'' + 2(\ln\sigma)'(\ln\rho)' - 2(\ln\rho)'^2 + 2(\ln\rho)''\big\}\,; \tag{8}$$

(ii) becomes

$$A = \sigma^2\big\{(\ln\sigma)'' - 2(\ln\sigma)'(\ln\rho)'\big\}\,; \tag{9}$$

(v) and (vi) become

$$A = \sigma^2\big\{(\ln\rho)'' - 2(\ln\rho)'^2\big\}\,. \tag{10}$$

The first two of these are equivalent to the pair of equations:

$$\begin{cases} \text{(a)} \quad A = \sigma^2(\ln\sigma)'' - 2\sigma^2(\ln\sigma)'(\ln\rho)' \\ \text{(b)} \quad\ 0 = -(\ln\rho)'^2 + (\ln\rho)'' + 2(\ln\sigma)'(\ln\rho)' \end{cases} \tag{11}$$

while the third becomes

$$\begin{aligned} A = \sigma^2(\ln\rho)'' - 2\sigma^2(\ln\rho)'^2 \quad &\overset{\text{(b)}}{\Longrightarrow} \quad \frac{A}{\sigma^2} = -(\ln\rho)'^2 - 2(\ln\sigma)'(\ln\rho)' \\ &\overset{\text{(a)}}{\Longrightarrow} -(\ln\rho)'^2 = (\ln\sigma)''\,. \end{aligned} \tag{12}$$

We can combine (11)(a) and the first identity of (12) to deduce

$$\begin{aligned} (\ln\rho)'' - (\ln\sigma)'' = 2(\ln\rho)'((\ln\rho)' - (\ln\sigma)') \;\Rightarrow\; \left(\ln\left(\frac{\rho}{\sigma}\right)\right)'' = 2(\ln\rho)'\left(\ln\left(\frac{\rho}{\sigma}\right)\right)' \\ \Rightarrow (\ln|(\ln u)'|)' = 2(\ln\rho)' \;\Rightarrow\; (\ln u)' = c\rho^2 \end{aligned}$$

for a constant c, where we have written $u = \rho/\sigma$. This determines σ as a function of ρ:

$$\frac{\rho}{\sigma} = (1/a)e^{\int c\rho^2 \mathrm{d}t} \quad \Rightarrow \quad \sigma = a\rho e^{-\int c\rho^2 \mathrm{d}t} \tag{13}$$

for constants a and c. It also yields the identity:

$$(\ln\rho)' - (\ln\sigma)' = c\rho^2 \quad \Longrightarrow \quad (\ln\sigma)'' = (\ln\rho)'' - 2c\rho\rho'\,.$$

When we combine this with the last identity of (12), we obtain

$$
\begin{aligned}
(\ln\rho)'' + (\ln\rho)'^2 = 2c\rho\rho' \quad &\Longrightarrow \quad \rho'' = 2c\rho^2\rho' \\
&\Longrightarrow \quad \rho' = \frac{2c}{3}\rho^3 + e
\end{aligned} \tag{14}
$$

for another constant e. Then from (13):

$$
\sigma = a\rho e^{-\int c\rho^2 \mathrm{d}t} = a\rho e^{-\int \frac{\rho''}{2\rho'}\mathrm{d}t} = a\rho e^{-\frac{1}{2}\ln|\rho'|+B} = b\rho|\rho'|^{-1/2}\,, \tag{15}
$$

for constants B and b where $A = \begin{cases} -3b^2 e & \text{if} \quad \rho' > 0 \\ +3b^2 e & \text{if} \quad \rho' < 0 \end{cases}$. Conversely, given a solution ρ to (14) with σ given by (13), equations (8), (9) and (10) are satisfied with $A = \begin{cases} -3b^2 e & \text{if} \quad \rho' > 0 \\ +3b^2 e & \text{if} \quad \rho' < 0 \end{cases}$. Specifically,

$$
(\ln\sigma)' = (\ln\rho)' - \tfrac{1}{2}\frac{\rho''}{\rho'} = (\ln\rho)' - c\rho^2 \;\Longrightarrow\; (\ln\sigma)'' = (\ln\rho)'' - 2c\rho\rho'
$$

and we now substitute.

Explicit solutions can be obtained by solving (14). In the case when $e = 0$, then up to an affine linear change in the t coordinate, the solution is given by $\rho(t) = t^{-1/2}$ with

$$
\sigma(t) = at^{-1/2}e^{\frac{3}{4}\int t^{-1}\mathrm{d}t} = at^{1/4}\,.
$$

This corresponds to an incomplete Ricci flat ($A = 0$) metric defined on the half-space $t > 0$.

In the case when $e \neq 0$, relabel the constants such that

$$
\rho' = \alpha(\rho^3 - \beta^3) = \alpha(\rho-\beta)(\rho^2 + \beta\rho + \beta^2) \quad (c = 3\alpha/2 \text{ and } e = -\alpha\beta^3)\,. \tag{16}
$$

Then

$$
\frac{\mathrm{d}\rho}{\alpha(\rho-\beta)(\rho^2+\beta\rho+\beta^2)} = \mathrm{d}t
$$

which can be integrated explicitly.

Lemma 5.1. (i) *For $\alpha < 0$ and $\beta > 0$, there is a solution $\rho(t)$ to* (16) *that exists for all $t \geq 0$, satisfying $\rho(0) = 0$, $\rho'(t) > 0$ for all $t \geq 0$ and $0 < \rho(t) < \beta$ for all $t > 0$. As $t \to \infty$, $\rho(t) \to \beta$ and $\rho'(t) \to 0$.*
(ii) *For $\alpha > 0$ and $\beta < 0$, there is a $t_0 > 0$ and a solution $\rho(t)$ to* (16) *that exists for all $t \in [0, t_0)$ satisfying $\rho(0) = 0$, $\rho'(t) > 0$ for all $t \in [0, t_0)$ and that tends to infinity as $t \to t_0^-$.*

Proof. (i) A solution $\rho(t)$ to (16) in a neighbourhood of $t = 0$ satisfying $\rho(0) = 0$ is guaranteed by the general existence theory of ODEs (see for example [5] (10.4.5)). Without loss of generality we can suppose that $\alpha = -1$ and $\beta = 1$ so the equation has the form

$$
\rho' = -\rho^3 + 1\,. \tag{17}
$$

Clearly $\rho'(t) > 0$ provided $\rho(t) < 1$. Suppose that $\rho(t)$ achieves the value 1 and let $t_0 > 0$ be the first time for which this occurs. Then from (17), $\rho'(t_0) = 0$. On differentiating (17), we see that $\rho''(t_0) = -3\rho^2(t_0)\rho'(t_0) = 0$, and so on; by

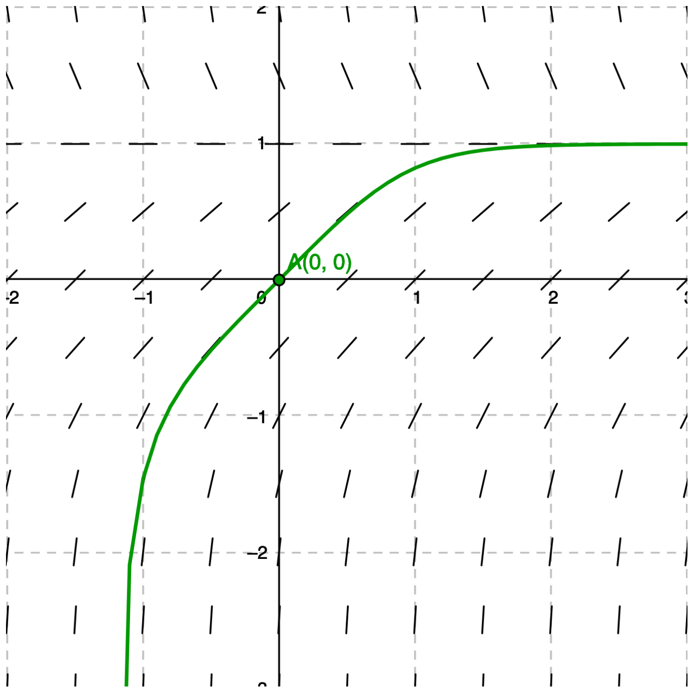

FIG. 1: Solution to (16) with $\alpha = -1$, $\beta = 1$ and $\rho(0) = 0$

recursion all derivatives $\rho^{(n)}(t_0) = 0$. But by analyticity of the solution (see [5] (10.5.3)), this means that $\rho(t) \equiv 1$ for all t, contradicting the initial condition $\rho(0) = 0$. Thus $\rho(t) < 1$ for all $t \geq 0$.

Clearly any interval of existence $[0, t_1)$ can be extended to $t \geq t_1$, so the solution exists for all time $t \geq 0$ with $\rho(t) \to 1$ and $\rho'(t) \to 0$ as $t \to \infty$.
(ii) Without loss of generality, suppose that $\alpha = 1$ and $\beta = -1$, so that (16) takes the form

$$\rho' = \rho^3 + 1\,. \tag{18}$$

This time we can appeal to the explicit equation determining ρ obtained on integrating (18) with $\rho(0) = 0$:

$$\frac{1}{3}\ln\frac{\rho+1}{|\rho^2-\rho+1|^{1/2}} + \frac{\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(\rho - \frac{1}{2}\right)\right) + \frac{\pi\sqrt{3}}{18} = t\,.$$

Then as $\rho \to \infty$, the left-hand side approaches $\frac{2\sqrt{3}\pi}{9}$ which yields the upper bound $t_0 = \frac{2\sqrt{3}\pi}{9}$. $\square$

In the following theorem, we consider *ends* as components of the complement of the set $\varepsilon \leq t \leq 1/\varepsilon$ for ε small.

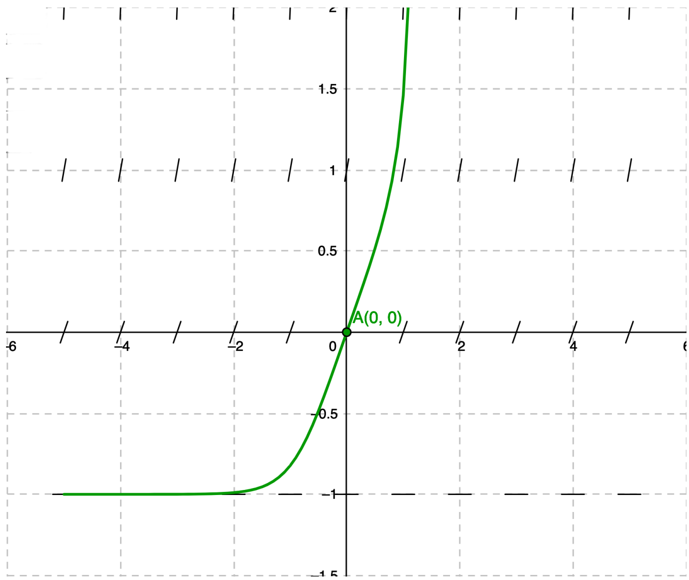

FIG. 2: Solution to (16) with $\alpha = 1$, $\beta = -1$ and $\rho(0) = 0$

Theorem 5.2. *Solutions to equation* (16) *yield two families of* 4*-dimensional Einstein metrics. Each member of the first family is a complete metric defined on the upper half space* $t > 0$*, having negative Ricci curvature and two ends: one asymtotic to hyperbolic* 4*-space and the other to* $\mathbb{R}^2$*. Each member of the second family is incomplete, defined on the space* $0 < t < t_0$ *for a fixed constant* t_0*, and has negative Ricci curvature.*

Proof. Consider the solutions to (16) given by Lemma 5.1(i) and as above, set $e = -\alpha\beta^3 > 0$. At $t = 0$, $\rho(0) = 0$, $\rho'(0) = e$, $\rho''(0) = \rho'''(0) = 0$. Thus the Taylor expansion about $t = 0$ has the form $\rho(t) = et + \mathcal{O}(t^4)$. For σ we have $\sigma(0) = 0$ and

$$\sigma = \frac{b\rho}{\sqrt{\rho'}} \Rightarrow \sigma' = \frac{b(\rho')^{3/2} - \frac{1}{2}b\rho(\rho')^{-1/2}\rho''}{\rho'} \Rightarrow \sigma'(0) = b\sqrt{e}\,.$$

Furthermore, $\sigma''(0) = \sigma'''(0) = 0$, so that about $t = 0$, we have $\sigma(t) = b\sqrt{e}t + \mathcal{O}(t^4)$. In particular, being of type $g_H := (\mathrm{d}t^2 + \mathrm{d}x_2{}^2 + \mathrm{d}x - 3^2 + \mathrm{d}x_4{}^2)/t^2$, for $t > 0$, the metric is complete in a neighbourhood of the boundary $t = 0$.

The Einstein constant can be deduced from (10), (16) and the expression (15) for σ, specifically $A = 3b^2\alpha\beta^3 < 0$.

In order to study the ends of the resulting Einstein manifold, we consider the exterior to the set $\varepsilon \leq t \leq 1/\varepsilon$ for ε small. As $t \to \infty$, then $\rho(t) \to \beta$, $\rho'(t) \to 0$ and $\sigma(t) \to \infty$. Thus the metric approaches an end of the form $\mathbb{R}^2$ with metric $(\mathrm{d}x_3{}^2 + \mathrm{d}x_4{}^2)/\beta^2$. Finally, the Taylor expansions of $\rho(t)$ and $\sigma(t)$ about $t = 0$ show that $g_H - g \to 0$ as $t \to 0^+$ (incorporating the constants into the coordinates), for example $\sigma(t)^2 = t^2 + \mathcal{O}(t^5)$ and $\dfrac{\mathrm{d}t^2 + \mathrm{d}{x_1}^2}{t^2 + \mathcal{O}(t^5)} - \dfrac{\mathrm{d}t^2 + \mathrm{d}{x_1}^2}{t^2} =$ $(\mathrm{d}t^2 + \mathrm{d}{x_1}^2)\mathcal{O}(t^5)/(t^4 + \mathcal{O}(t^7)) \to 0$ as $t \to 0^+$ which shows asymptotic convergence to g_H.

A similar analysis takes place for the solutions to (16) given by Lemma 5.1(ii), but this time $\rho(t) \to \infty$ as $t \to t_0^-$, showing the incompleteness of the metric. □

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